Is the optimal implementation inefficient? Elementarily not.*

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Abstract

Sharing graphs are a local and asynchronous implementation of lambda-calculus beta-reduction (or linear logic proof-net cut-elimination) that avoid useless duplications. Empirical benchmarks suggest that they are one of the most efficient machinery, when one wants to fully exploit the higher-order features of lambda-calculus. Although, we still lack confirming grounds with theoretical solidity to dispel uncertainties about the adoption of sharing graphs. Aiming at analysing in detail the worst-case overhead cost of sharing operators, we restrict to the case of elementary and light linear logic, two subsystems with bounded computational complexity of multiplicative exponential linear logic. In these two cases, the bookkeeping component is unnecessary, and sharing graphs are simplified to the so-called "abstract algorithm". By a modular cost comparison over a syntactical simulation, we prove that the overhead of shared reductions is quadratically bounded with respect to cost of the naive implementation. This result generalises and strengthens a previous complexity result, and implies that the price of sharing is negligible, if compared to the obtainable benefits on reductions requiring a large amount of duplication.

1998 ACM Subject Classification F.1.1 Models of Computation, F.1.3 Complexity Measures and Classes, F.4.1 Mathematical Logic.

Keywords and phrases optimality, sharing graphs, λ-calculus, complexity, linear logic.

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

1.1 Intelligence of sharing graphs

Redundancy and non-locality An ideal implementation of a functional programming language aims at satisfying two properties: sharing, i.e. to avoid the duplication of work, locality, i.e. to be parallelisable on architectures with multiple computing agents. The presupposal for both is a fine-grain implementation of material duplication.

Consider for instance the β-redex $T = (\lambda x.M)(\lambda y.N)$ and assume that x occurs $k+1$ times in M. In the λ-term $M\{^{\lambda y.N}/_x\}$ obtained by reducing it, we have k new copies of N, and then, k additional copies of any redex in it. Also, such a reduction step cannot be fired in parallel with any reduction in N. To solve these kind of problems, we may consider to switch to a graph rewriting approach. Graph reduction represents the state-of-the-art implementation model for lazy languages [20], and its essence dates back to the early seventies [23]. The

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* Preliminary results of this work were previously presented [14] with a stronger yet unsolved claim.
† Work mainly carried out during: PhD studentship at Paris 13 and Bologna, ATER at Paris 7.

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

key idea can be formalised with proof-nets of linear logic (LL), where T can be expressed as in Figure 1a. With respect to the syntax tree of T we notice: the explicit connection of the λ-link with its variable x; a ?-link connecting the $k+1$ occurrences of x in M; a box (the dotted square) around the argument of the application (the @-link), to mean $\lambda y.N$ is a duplicable term. Such an information can be equivalently formalised by associating the boxing depth to each link—e.g. the rightmost λ-link has index 1. Now, a linear-β reduction step rewrites Figure 1a in Figure 1b. This delays the duplication of the box, which is now shared, and allows performing reduction in N independently from M. Unfortunately, this is still quite unsatisfactory. Indeed, if an occurrence of x appears in argument position within M, we get a sort of "virtual β-redex" between the @-link above this occurence of x, and the λ-link of $\lambda x.N$. In order to reduce such a redex, the only option is to make it explicit by firing the exponential redex between the contraction ?-link and the promotion !-link. But this would break the sharing of the box, by duplicating it with a non-local rewriting step.

Sharing and locality Sharing graphs [18, 13, 4] include instead, by their very design, the solution to these problems of duplication. Back to our example, instead of performing the duplication of the lowermost box, we can apply the "triggering" rule (t) shown in Figure 1b-c. This introduces a link of a new kind $|{+}\rangle$, called *mux* (multiplexer), that has $k+1$ premisses and index 1 as the content of the box. Muxes perform duplications of boxes in a local, link-by-link way, following the approach introduced by interaction nets [17]. For instance, the $|{+}\rangle$-link in the example can duplicate the λ-link only, by the $(d^{\pm}_{\multimap})$ rule in Figure 1c-d. This allows N being kept shared, whilst copies of the λ-link can now independently interact with @-links in M. Also, the $(d^{\pm}_{\multimap})$ rule originates a sort of co-sharing link, the *negative mux* with kind $|{-}\rangle$ and the same index, which implements sharing of contexts (outputs), instead of terms (inputs). In addition to the case of λ-links, there is one duplication for each other link $(@, !, ?)$, but it is applicable only when the mux faces its principal port, i.e. its conclusion, depicted by an outgoing arrow. This makes duplication *lazy*—it is performed only when a mux obstructs the formation of other redexes. The life of a $|{+}\rangle$-link may end in two ways: by being merged into it, when it reaches the premiss of a ?-link, or by annihilating with a facing $|{-}\rangle$-links (i.e. the facing pair of links reduces to the identity). But positive muxes may also need to swap with negative ones, i.e. they duplicate each other. Therefore, we mark muxes with the index inherited from the exponential redex, e.g. 1 in our running example, so we can distinguish two kinds of redexes with two opposite muxes: annihilation when indices are equal, swap when they are different. In general, we would also need a supplemental mechanism or information tracking, the so-called "oracle", which manages indexes, as a local implementation of digging and dereliction of LL. But this is not the case of proof-nets for terms typed in the elementary and light variants of LL (ELL and LLL) [12, 6]. They are obtained by a restriction on usual exponential boxes that makes indices immutable by definition. Hence, for their sharing implementation we can simply consider so called *abstract algorithm* of sharing graphs (ASG). Thanks to muxes, the sharing graph G that is the normal form of T is a (possibly enormously) compressed representation of the proof-net $\overline{T}$ we would have obtained by ordinary cut-elimination. To retrieve $\overline{T}$ from G, we need the *read-back* (RB)—a set of additional rewriting rules for sharing graphs, which unlatch muxes to let them freely duplicate downwardly, i.e. from the root of the term to its inputs.

▷ This paper considers elementary proof-nets (EPN) to represent ELL and LLL typed terms, or proofs (presented in § 2), and their sharing implementation with ASG, including RB (§ 3). Since the weakening rule in EPN and the garbage collector in ASG would introduce technical overhead and little interest, they will be excluded by restricting to terms of λI-calculus.

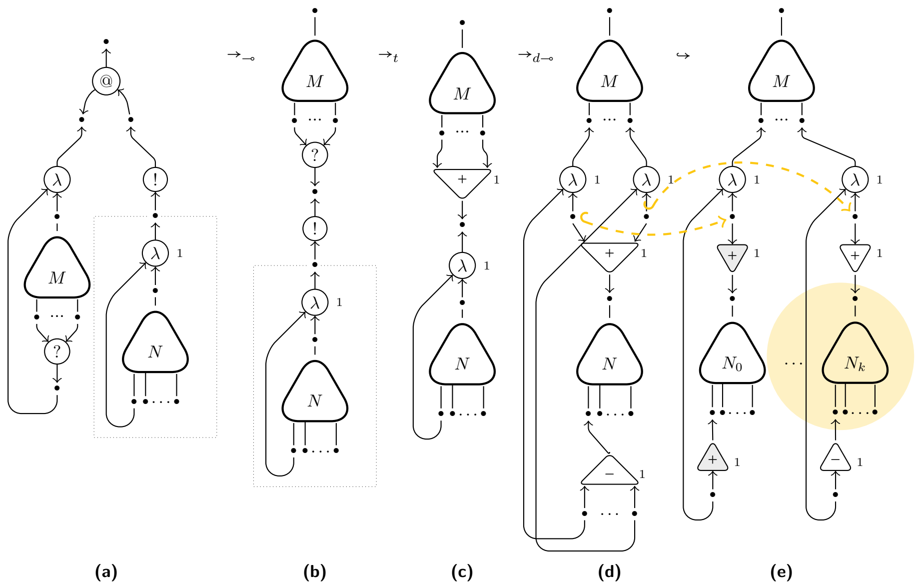


Figure 1 Examples: sharing reductions (a-d) on a proof-net; unshared graph (e) unfolding of d.

1.2 Efficiency of sharing graphs

Question The possible benefits of sharing were astonishingly evident from the very first sequential implementations of sharing graphs. For example, in the normalisation of λ-terms benchmarks of BOHM [3] recorded polynomial times, against traditional languages (Caml Light and Haskell) [4, Ch. 10] requiring exponential times. But sharing and locality may come with a price. *What is the price of sharing graphs?* To answer this question, very recently broadly surveyed by Asperti [1], we first need the notion of cost. We cannot use the number of β steps, since, even though they are a reasonable measure for the λ calculus [9], since in sharing graphs whole families of redexes are reduced simultaneously, and this is the reason why they realise the Lévy optimal reduction [19]. Nor we can use the number of such parallel β-steps, since it would be an enormously parsimonious measure. Indeed, a polynomial number of family reductions in the size s of a term may hide a concrete cost bigger than a tower of exponentials of s, caused by the oracle rules [5] or more simply by the local duplication rules [2]. In conclusion, a study of the complexity of sharing graphs is necessarily limited by the state of the art to take the form of a comparison to some existing reduction system, i.e. by the approach of "ICC in the small" [10], and to use a cost measure based on standard rewriting theory—counting the size of rewriting at each step.
▷ In § 4.1 we shall define two cost functions $\mathcal{C}_{\mathsf{ASG}}$ and $\mathcal{C}_{\mathsf{EPN}}$, respectively for the ASG and EPN reductions.

A first partial answer The only previous contribution which tackled the complexity study to ASG is [7]. In which, Baillot, Coppola and Dal Lago restricted to the case of ELL and LLL, and explicitely exploited that in (the affine variants of) ELL and LLL, the cost of the cut-elimination of a proof-net N with maximum boxing index d is related respectively by a Kalmar elementary function in the size of N and rank d, in ELL, or by a polynomial function in the size of N and degree d, in LLL. By means of a quantitative semantical tool [8] inspired by the geometry of interaction [11], the authors proved that the cost of a normalisation with sharing graphs of ELL and LLL proof-nets reamins in the two aforementioned complexity classes. Buth, teyr were not able to give any explicit bound to the overhead that the sharing graphs migght introduce in the worst case—for instance, when becomes less effective, since the reduction dose not require a relevant amount of duplications. Moreover, the technical approach hardly seems adaptable to prove a similar result for the more general case featuring the "oracle".

1.3 A syntactical bound to the sharing overhead

Unshared simulation The natural correspondence between sharing graphs and proof-nets lies in a *simulation* relation on which our complexity study is founded. Given a proof-net N, any ASG reduction sequence σ on N can be simulated by a EPN sequence ρ on the same N. Since the only difference between the two rewriting systems is the style of duplication, the resulting graphs $\sigma(N)$ and $\rho(N)$ may differ for the number of copies of some subgraphs: for instance, a redex R in $\sigma(N)$ may have been duplicated several times by ρ into a set R' of redexes of $\rho(N)$; the reduction of R in $\sigma(N)$ can then be simulated by reducing any redex of $\rho(N)$ in R'. Such an intuition can be detailed and understood by employing an intermediate reduction system, called *unshared graphs* [16]. Their key feature is the fact that the exponential redex as in Figure 1b would be rewritten in $k+1$ copies of the box (as one usually does on proof-nets), plus $k+1$ unary muxes (instead of one k-ary mux of Figure 1c). These links are called *lifts*, and play the mere role of markers for the presence of muxes in corresponding sharing graphs. In particular, they can propagate along the graph, but they do not affect any other link. Indeed, by erasing lifts we simply obtain a proof-net. For instance, the $(d\lambda)$ step in Figure 1c-d, can be simulated by $k+1$ propagation steps that reaches Figure 1e, that is the *unfolding* of Figure 1d. Yellow arrows illustrate such relation. ▹ Formal definitions of unshared graphs and their simulation are provided in § 4.2.

Quantitative unshared simulation We will bring the unshared simulation up to a quantitative level, by introducing a labelling of lifts. Given an unshared graph U being the unfolding of a sharing graph G, we will build the *sharing context* of a vertex v as the sequence of lift labels that are along its access path from the root of U. Its sharing context allows us to understand if v is shared in G (equivalently: "has v been previously copied in U"?), and how much is so ("how many other copies of v are in U"?). The set of such shared objects is called the *share*, and conceptually represents the subtraction of the graph of G from that of U. The pale yellow circle highlights the share in Figure 1e, which does not include N_0 as we interpret it as the *master copy* (the stereotype) with respect of the other k. Therefore, given two reduction sequences on the same proof-net, σ of ASG and ρ of EPN, we are able to compare $\mathcal{C}_{\mathsf{ASG}}(\sigma)$ to $\mathcal{C}_{\mathsf{EPN}}(\rho)$. Indeed, looking at variations in the size of the share, we can "read" both of the two costs in the intermediate UG reduction, and decompose our analysis. The portion of $\mathcal{C}_{\mathsf{ASG}}(\sigma)$ caused by logical, duplicating, merging and annihilation rules, can be linearly bounded to $\mathcal{C}_{\mathsf{EPN}}(\rho)$ by a mere local observation. Swap rules are instead quadratically bounded to $\mathcal{C}_{\mathsf{EPN}}(\rho)$. Hence, the overhead of ASG with respect to its EPN simulation admits a quadratic bound.

▸ **Theorem 1** (Complexity comparison). *Let $N, N' \in EPN$, $G \in ASG$ such that $N \xrightarrow{\bar{\sigma}}{}^{*}_{ASGR} G$ and $N \xrightarrow{\bar{\rho}}{}^{*}_{EPN} N'$, where $\bar{\rho} \rightarrowtail \bar{\sigma}$. Then $\mathcal{C}_{ASG}(\bar{\sigma}) \leq q\,(\mathcal{C}_{EPN}(\bar{\rho}))$ where q is a quadratic function.*

The idea of the proof we have just illustrated will be developed in § 4.3, where details of technical notions and statements of intermediate steps will be given. Proofs of these former are instead omitted for the lack of space, but can be found in the appendix.

Contributions We study the complexity of reducing (proof-nets representing) λ-terms typed in the elementary or light linear logic and relate it to the cost of the naive reduction. In particular, we give a worst-case bound of the cost of sharing reduction as a quadratic function in the cost of the naive reduction. Three are the axis along which this paper improves [7]: **1.** *Strength.* We give a clear bound to the overhead of sharing reduction. **2.** *Generality.* Our analysis considers strategy-agnostic reductions of arbitrary length, instead of normalisation, and includes the read-back rules (which are sub-optimal). In this way we get a more uniform, and perhaps fairest, complexity comparison. **3.** *Scalability.* The technical approach bases on a quantitative extension of an elegant syntactical simulation between sharing graphs and proof-nets; this provides modularity for our complexity analysis, and appears to give room for further investigations about more general cases also.

2 Intuitionistic elementary and light logics

We start by introducing proof-nets of ELL and LLL. We define first the two intuitionistic and weakening-free logics by a levelled sequent calculus inspired by the approach of Guerrini, Martini and Masini [15], which represents a typing system for λI terms. Then, we give the translation of sequent proofs in levelled proof-nets, that are directed hypergraphs where every (occurrence of a) formula is a vertex and every inference rule is a directed hyperedge (called link) that goes from the premisses of the rule to its conclusions. Axioms and cuts correspond instead to a direct plugging of the two sub-proofs. Finally, we will present cut-elimination as reduction of proof-nets.

2.1 Sequent calculus and typed lambda terms

▸ **Definition 2** (Logics and typing). Given L a set of *literal* symbols, the *formulas* of ELL and LLL are built from the following grammar.

$$T ::= L \mid T \multimap T \mid !T \mid \S T. \tag{1}$$

A *levelled formula* is a pair T^n where T is a formula and $n \in \mathbb{N}$. Given a set V of *variables*, the set of *terms* is defined from the standard definition of the Λ calculus:

$$t ::= V \mid \lambda V.t \mid t\,t, \tag{2}$$

by constraining abstraction so that it bounds at least one occurrence (i.e. x must appear as a free variable of t in order to write $\lambda x.t$). The *sequent calculus* of the *first-order, weakening-free, intuitionistic fragment of ELL and LLL* is defined in Figure 2 and represents a type assignment system for the Λ_I calculus.

▸ Remark 3 (LLL ⊂ ELL). Observe that § cannot appear in ELL formulas and that the light version of the (!) rule is an instance of the elementary one. Moreover, any proof π of LLL can be encoded in ELL by using the logical modality ! instead of §. For this reason and the sake of simplicity, in the rest of the paper we shall refer only to the elementary version.

2.2 Proof-nets

▸ **Definition 4** (Elementary proof-nets). Given a set of *vertices*, a *link* is a directed hyperedge, i.e. a triple $(V\ \kappa^n\ u)$, where: u is a vertex called *conclusion*, V is a non empty sequence of vertices called *premisses*, κ is a label called *kind*, $n \in \mathbb{N}$is called *level*. Depending on the kind of a link l, a *polarity* is assigned to each of its vertices, i.e. an element of $\{\iota, o\}$

$$\frac{}{x : A^n \ \vdash \ x : A^n}\ (\mathrm{Ax}) \qquad \frac{\Delta_1^{n+1}, \Gamma_1^n \ \vdash \ t : A^n \qquad \Delta_2^{n+1}, \Gamma_2^n, x : A^n \ \vdash \ u : B^n}{\Delta_1^{n+1}, \Delta_2^{n+1}, \Gamma_1^n, \Gamma_2^n \ \vdash \ u[{}^t/_x] : B^n}\ (\mathrm{Cut})$$

$$\frac{\Delta^{n+1}, \Gamma^n, \{x : A^n\} \ \vdash \ t : B^n}{\Delta^{n+1}, \Gamma^n \ \vdash \ \lambda x.t : A \multimap B^n}\ (\multimap) \qquad \frac{\Delta_1^{n+1}, \Gamma_1^n \ \vdash \ t : A^n \qquad \Delta_2^{n+1}, \Gamma_2^n, x : B^n \ \vdash \ u : C^n}{\Delta_1^{n+1}, \Delta_2^{n+1}, \Gamma_1^n, \Gamma_2^n, y : A \multimap B^n \ \vdash \ u[{}^{yt}/_x] : C^n}\ (\bar{\multimap})$$

(a) Common rules: axiom, cut, abstraction, application

$$\frac{\Gamma^{n+1} \ \vdash \ t : A^{n+1}}{\Gamma^{n+1} \ \vdash \ t : !A^n}\ (!)$$

(b) Elementary promotion

$$\frac{\{x : A^{n+1}\} \ \vdash \ t : A^{n+1}}{\{x : A^{n+1}\} \ \vdash \ t : !A^n}\ (!) \qquad \frac{\Gamma^{n+1}, \Delta^{n+1} \ \vdash \ t : A^{n+1}}{\S\Gamma^n, \Delta^{n+1} \ \vdash \ t : \S A^n}\ (\S)$$

(c) Light promotion – Γ, Δ may be empty

$$\frac{\Delta^{n+1}, \Gamma^n, (x_i : A^{n+1})_{1 \le i \le m} \ \vdash \ t : B^n}{\Delta^{n+1}, \Gamma^n, y : !A^n \ \vdash \ t[{}^y/_{x_i}]_{1 \le i \le m} : B^n}\ (?)$$

(d) Contraction

Figure 2 Levelled sequent calculus of ELL and LLL

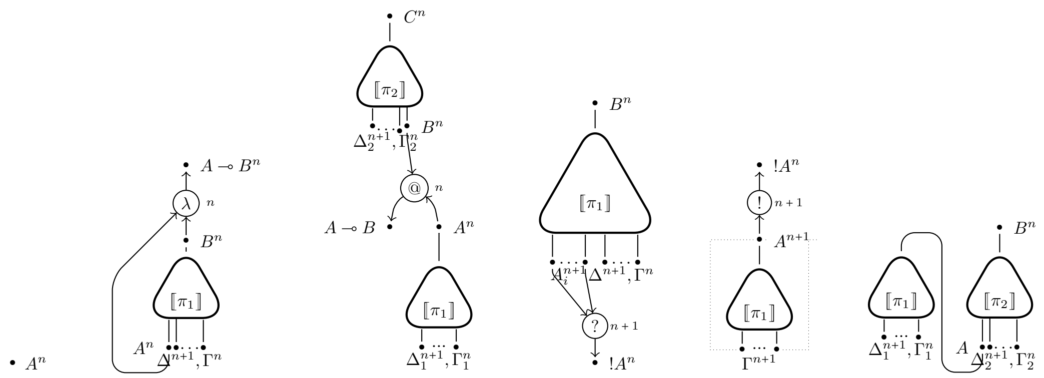
(input and output), and the *arity* may be fixed, i.e. the number of vertices of l. In order to keep a better correspondence with the underlying λ-calculus, the kinds of links will correspond to the syntactical construct associated to the rule and not to the corresponding logical connective. Also, links are depicted so that positioning follows the input/output computational interpretation, o above, ι below; whilst arrows orientation follows the proof-theoretical viewpoint, premisses inward, conclusions outward.

A proof-net of the fragment of ELL of our interest, to which we shall simply refer as an *elementary proof-net* (EPN), is the hyper-graph N obtained by the *translation* $[\![\pi]\!]$ of some sequent proof π, as inductively defined as follows. Let R be the last rule of π, assume its shape to be as in one of Figures 2a, 2b and 2d. Also, let π_1, π_2 respectively be the sub-proofs of π (if any) whose conclusions are the leftmost and rightmost premisses of R. Then the translation of π is given by the case analysis in Figure 3. There we assume that $[\![\pi_1]\!], [\![\pi_2]\!]$ are disjoint, and also that, except when otherwise depicted, both are disjoint from the vertices introduced by the inductive steps (i.e. new vertices are “fresh”). Although we shall label vertices with their names, in the picture we used formulas to ease the reading and stress the correspondence with the sequent proof.

The *conclusions* of $[\![\pi]\!]$ are the vertices corresponding to formulas appearing in the sequent proved by π — *input* vertices stand for formulas on the left-hand side of the sequent, *output* ones for those on the right-hand side. The set of the conclusions of $N \in$ EPN is called its *interface* and denoted by $\mathsf{iface}(N)$; any vertex of N not in $\mathsf{iface}(N)$ is an *internal vertex* of N, and the corresponding set is denoted by $\mathsf{int}(G)$.

▸ Remark 5 (λ-terms). The input-output relation allows to associate a λ-term to a proof-net—if we erase exponential links, we essentially obtain the syntax tree of the term.

▸ **Definition 6** (Boxes). The level of a vertex v is denoted as $\ell(v)$, and we shall write the same for a link, meaning the level of its premiss(es). The maximum level of links in N is written $\partial(N)$. A *box* of level n in $N \in$ EPN is a sub-graph B of N whose links and vertices have level not smaller that n and that is maximal with respect to inclusion. The ι conclusion v of B is called its *principal door*, whilst the o conclusions $u_1, \ldots, u_k$ *auxiliary doors*. We shall then denote B as $(u_1, \ldots, u_k \ \langle\!\langle B \rangle\!\rangle \ v)$. Boxes are depicted with dotted squares.



(a) Axiom **(b)** Abstraction **(c)** Application **(d)** Contraction **(e)** Promotion **(f)** Cut

Figure 3 Elementary proof-nets, as translated from sequent proof. See Definition 4 and Fig. 2.

▸ **Definition 7** (Paths)**.** A *path* from u_0 to u_k in a graph G is a sequence of vertices $(u_0, \dots, u_k)$ for which there is a sequence of links $l_0, \dots, l_{k-1}$ such that $u_i \neq u_i + 1$ and u_i belongs to both the vertices of l_i and those of l_{i+1}, for any $0 \le i < k-1$. A *downward path* is a path such that: u_i is not the first premiss of a $\multimap$-link and is an o-vertex of l_i; and u_{i+1} is an ι-vertex of l_i, for any $0 \le i < k-1$. If these holds, then $(u_k, \dots, u_0)$ is an *upward path* from u_k to u_0. We shall write $u \sim v$ when there is a path from u to v, $u \rightsquigarrow v$ when there is a downward one, $v \leftsquigarrow u$ when $u \rightsquigarrow v$. A *rooted* path is a downward path starting from the o-conclusion of G.

▸ Remark 8 (Boxes). Notice that auxiliary doors of a box B are always premisses of an exponential link. and that B is always connected: there exists a path between any two of its vertices. Also, boxes properly nest: given two distinct boxes, either they are completely disjoint, or one is included into the other. Our definition of box is slightly different from the standard one, as it is does not include the exponential formulas, but just take the interior of the box.

▸ **Definition 9** (EPN reduction)**.** The rewriting relation $\to_{\mathsf{EPN}}$, which implements the cut-elimination on proof-nets, is obtained by the context closure of the reduction rules $(\overset{+}{\multimap}), (D)$ respectively defined in Figures 4a and 4b.

▸ **Notation 10.** Reductions steps are denoted by Greek letters (e.g. ρ), sequences are marked with overlines ($\overline{\rho}$), reducts are denoted by functional notations ($N \to \rho(N)$).

▸ **Proposition 11** (Stratification)**.** *In EPN, levels are preserved by reduction.*

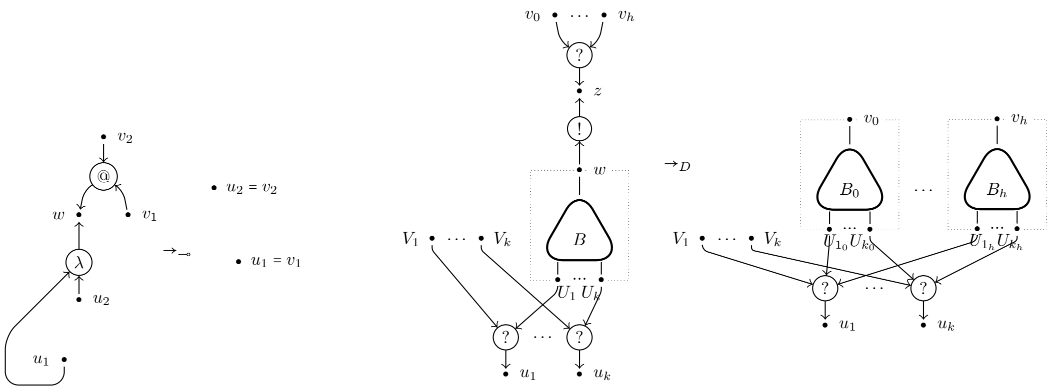
3 Abstract sharing graphs

In this section we introduce abstract sharing graphs (ASG) and their reduction, and recall the most important qualitative properties as an implementation of EPN.

3.1 Syntax and computation

▸ **Definition 12** (Sharing and read-back reductions)**.** The *sharing reduction* is the graph-rewriting relation $\to_{\mathsf{ASG}}$ given by the context closure of the following reduction rules.

Logics $(\multimap), (!), (t)$, defined in Figures 4a and 5a. On unary contractions, $(!) = (D)$.



(a) Multiplicative **(b)** Exponential

Figure 4 EPN reduction

Duplication $(d\overset{+}{\multimap}), (d\overset{-}{\multimap}), (d!), (d?)$, respectively defined in Figures 5c to 5f.
Mux interaction $(a), (s)$, defined in Figure 5b.

The *read-back reduction* $\rightarrow_{\mathsf{RB}}$ is obtained from the mux interaction rules and the followings.

Read-back duplication $(r\overset{+}{\multimap}), (r?), (m)$, defined in Figures 5g and 5h.

The RB-normal form of a graph G is called its *read-back* and written $\mathcal{R}(G)$. The reduction $\rightarrow_{\mathsf{ASGR}}$ is the union of $\rightarrow_{\mathsf{ASG}}$ and $\rightarrow_{\mathsf{EPN}}$. The set ASG of abstract sharing graphs is obtained by the closure of EPN with respect to $\rightarrow_{\mathsf{ASGR}}$.

3.2 Implementation of EPN

Sharing graphs with ASG and RB reductions represent a well-behaved rewriting system. $\rightarrow_{\mathsf{ASG}}$ is locally confluent. $\rightarrow_{\mathsf{RB}}$ and $\rightarrow_{\mathsf{ASGR}}$ are confluent. All three are strongly normalising, and the last two have normal forms in EPN [16, Thm.s 4, 11.i and 11.ii (for MELL)]. The traditional way of normalising proofs or terms with sharing graphs maximises the amount of sharing by postponing duplication as much as possible, thus performing first an ASG-normalisation and then an RB one. This gives a correct implementation of EPN reduction and a complete implementation of EPN normalisation.

▶ **Theorem 13** (Correctness). *If $N \in \mathsf{EPN}, G \in \mathsf{ASG}$ and $N \rightarrow^*_{\mathsf{ASG}} G$, then $N \rightarrow^*_{\mathsf{EPN}} \mathcal{R}(G)$.*

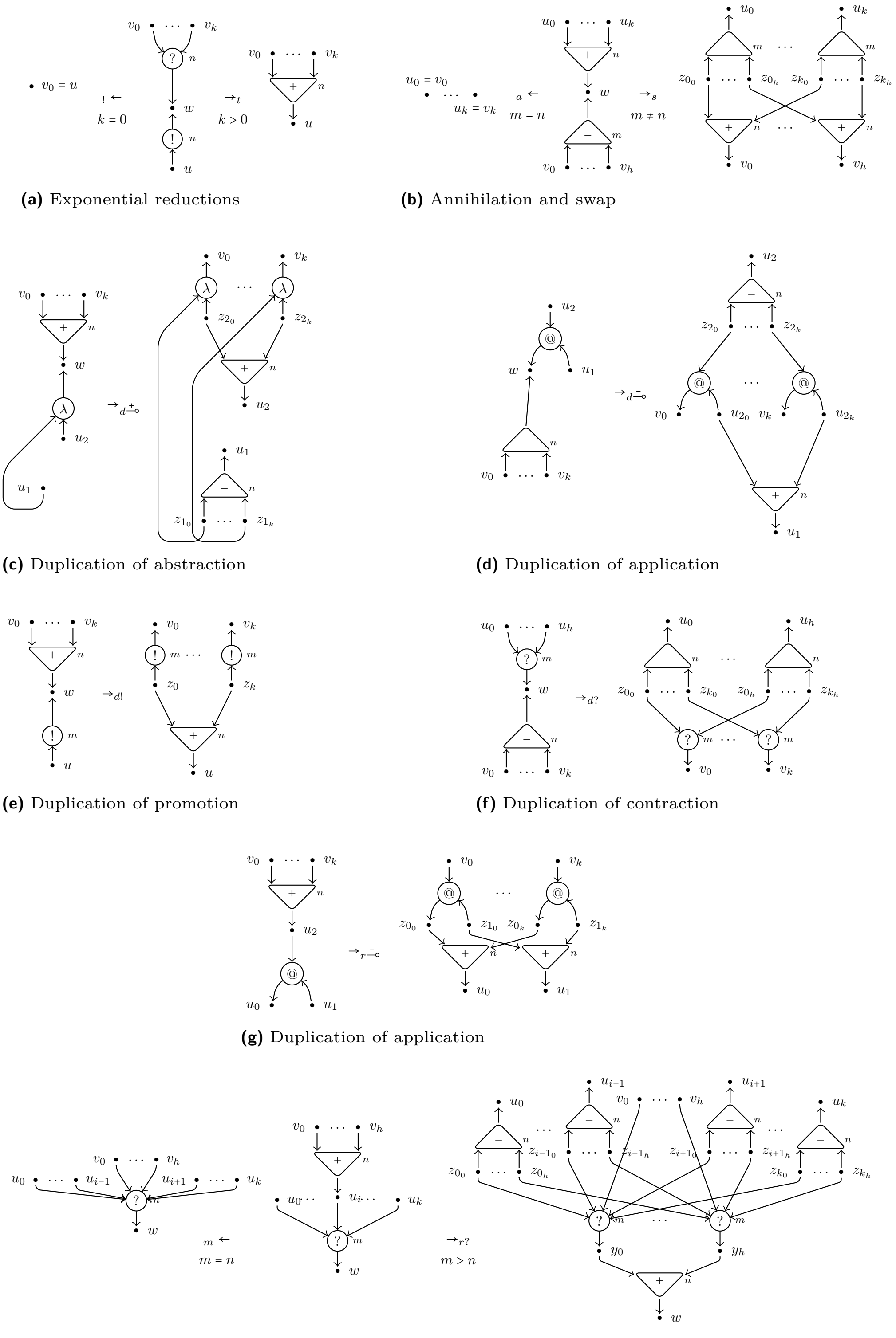
Proof. We refer to the original proof for λ-calculus [13], or the more syntactic one for MELL [16, Thm. 13]. ◀

▶ **Theorem 14** (Normalisation completeness). *If $N, \bar{N} \in \mathsf{EPN}$ with $\bar{N}$ normal, then there is $\bar{G} \in \mathsf{ASG}$ being ASG-normal and such that $N \rightarrow^*_{\mathsf{ASG}} \bar{G} \rightarrow^*_{\mathsf{RB}} \bar{N}$.*

Proof. See Asperti and Guerrini [4, Thm. 7.9.3.ii]. ◀

If instead we consider $\rightarrow_{\mathsf{ASGR}}$, sharing graphs can be shown to be more generally complete with respect to the whole EPN-reduction (not just normalisation). It suffices to prioritise RB redexes, thus enforcing exhaustive duplications of boxes.

▶ **Theorem 15** (Completeness). *For any $N, N' \in \mathsf{EPN}$ if $N \rightarrow_{\mathsf{EPN}} N'$ then $N \rightarrow^+_{\mathsf{ASGR}} N'$.*



(a) Exponential reductions

(b) Annihilation and swap

(c) Duplication of abstraction

(d) Duplication of application

(e) Duplication of promotion

(f) Duplication of contraction

(g) Duplication of application

(h) Merge with, or duplication of, contraction

Figure 5 ASG and RB reduction rules

Rule	$\mathcal{C}_{\mathsf{EPN}}(\rho)$
$(\multimap)$	9
(D)	$k \times \#B + 2k + 4$

(a) Classic reduction rules. In the case of (D), $k+1$ is the number of premisses of the ?-link, and B is the box enclosed by the !-link.

Rule	$\mathcal{C}_{\mathsf{ASG}}(\sigma)$
$(\multimap)$	9
$(!)$	6
(t)	$j+4$
$(d!)$	$3k$
$(d\overset{+}{\multimap}), (d\overset{-}{\multimap}), (r\overset{-}{\multimap})$	$5k$
$(d?), (r?)$	$(2j+3) \times k$
$(a), (m)$	k
(s)	$k \times l$

(b) Sharing and read-back rules. $j+1$, $k+1$ and $l+1$ respectively are the number of premisses of the ?-link, the first and the second $|\!\ast\rangle$-link, where involved.

Table 1 Costs assigned to classic and sharing reductions.

▸ Remark 16 (Optimality). Once equipped with a "call-by-need" needed strategy, the number of logical steps performed by ASG is minimised so the reduction reaches Lévy-optimality [16, Thm. 14 (for MELL)] [4, Thm. 5.6.4 (for λ-calculus)]. For the sake of generality, our focus will not be limited to the optimal strategy, and we shall analyse sharing graphs with the greatest strategy-agnosticism.

4 Computational complexity

We define two cost functions $\mathcal{C}_{\mathsf{ASG}}$ and $\mathcal{C}_{\mathsf{EPN}}$, respectively for the ASG and EPN reductions. Then, we introduce unshared graphs (UG), which allows us to bridge a simulation of ASG into EPN. By extending it with quantitative precision, we can compare within UG the complexity of the two reduction systems, and obtain that $\mathcal{C}_{\mathsf{ASG}}$ of a reduction is quadratically bounded in $\mathcal{C}_{\mathsf{EPN}}$ of its simulation.

4.1 Cost measures

▸ **Definition 17** (Size and variations)**.** The *size* of a graph G, written $\#G$, is the sum of the cardinality of the set of its vertices and the sum of the arities of its links. We remark that for a box $(u_1, \ldots, u_k \langle\!\langle B \rangle\!\rangle\, v)$, all of its doors, principal and auxiliary ones, belong to the sub-graph B and are accounted by $\#B$. Given M a metric on a graph G, e.g. the size, and ρ a reduction step, $\Delta c(\rho)$ denotes $M(\rho(G)) - M(G)$.

▸ **Definition 18** (EPN-reduction cost)**.** The cost $\mathcal{C}_{\mathsf{EPN}}(\rho)$ of a EPN-reduction step ρ on a levelled proof-net N is defined as the size of the symmetric difference between the vertices and links of N and those of $\rho(N)$. Namely, the cost of a given rule is computed in Table 1a. The cost of a reduction sequence $\bar{\rho}$ is the sum of the costs of each step which is composed of.

▸ **Definition 19** (ASG- and RB-reduction costs)**.** The cost $\mathcal{C}_{\mathsf{ASG}}(\sigma)$ and $\mathcal{C}_{\mathsf{RB}}(\sigma)$ of a ASG- or RB-reduction step σ is given in Table 1b.

4.2 Unshared simulation

4.2.1 Unshared graphs

▸ **Definition 20** (Unshared reduction and graphs)**.** The *unshared reduction* is the rewriting relation $\rightarrow_{\mathsf{UG}}$ obtained from $\rightarrow_{\mathsf{ASG}}$ by replacing the (t) rule with (tD), defined Figure 6 The *unshared read-back* $\rightarrow_{\mathsf{UR}}$ is just an alias of RB-reduction. The union of these two relations is written as UGR. An *unshared graph* UG-graph for short, is either a levelled proof-net, or the reduct of an unshared graph via UG- or UR-reduction.

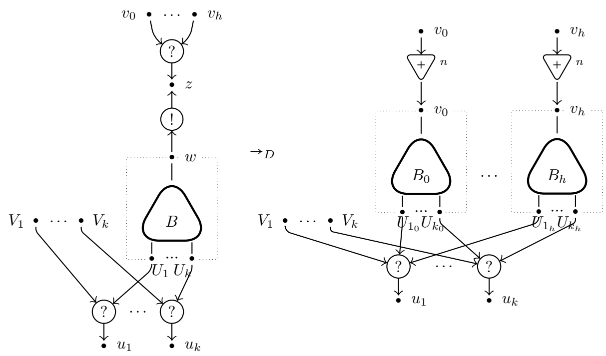


Figure 6 Duplication and triggering rule (where $h > 1$).

4.2.2 Unfolding and simulating sharing graphs in unshared graphs

▸ **Definition 21** (Sharing morphism). A *sharing morphism* $\mathcal{M}$ is a surjective homomorphism from UG to ASG that preserves the kind and level of links. We say that $G \in$ ASG *unfolds* to $U \in$ UG, written $G \hookrightarrow U$, if there is a sharing morphism $\mathcal{M}$ such that $\mathcal{M}(U) = G$. We shall use the same notation to relate vertices and links: if $v \in V(G)$ and $W \subseteq V(U)$ we write $v \hookrightarrow W$ to mean $\mathcal{M}(W) = v$, while if $m \in L(G)$ and $N \subseteq L(U)$ we write $m \hookrightarrow N$ when $\mathcal{M}(N) = m$.

▸ **Lemma 22** (Unfolded simulation). *For any $N \in$ EPN, $G \in$ ASG, if $N \xrightarrow{\bar{\sigma}}{}^{*}_{ASG} G$ then there exists $U \in$ UG such that $N \xrightarrow{\bar{\mu}}{}^{*}_{UG} U$ and $G \hookrightarrow U$. Moreover, for any $G' \in$ ASG, if $G \xrightarrow{\bar{\sigma}'}{}^{*}_{RB} G'$ then there exists $U' \in$ UG such that $U \xrightarrow{\bar{\mu}'}{}^{*}_{UR} U'$ and $G' \hookrightarrow U'$. We call $\bar{\mu}$ and $\bar{\mu}'$ the* unfolded simulations of $\bar{\sigma}$ *and $\bar{\sigma}'$, respectively, and write $\bar{\sigma} \hookrightarrow \bar{\mu}$ and $\bar{\sigma}' \hookrightarrow \bar{\mu}'$.*

4.2.3 Simulating sharing graphs into proof-nets

▸ **Definition 23** (Lift erasure). The *lift erasure* is the function $\mapsto$ that maps a $U \in$ UG to the $N \in$ EPN obtained by equating any two vertices u, v for which there is $(u \mid\!\ast\rangle\, v) \in L(U)$. Such function is extended to let it map vertices and links of U to those of N.

▸ **Lemma 24** (Lift erasure simulation). *For any $N \in$ EPN, $U \in$ UG, if $N \to^{*}_{UG} U$ then there is a unique $\bar{\sigma} : N \to^{*}_{EPN} N'$ such that $U \mapsto N'$.*

▸ **Definition 25** (Sharing implementation). Given $N \in$ EPN, $G \in$ ASG, we say N is *implemented by G*, written $G \rightarrowtail N$, if there is $U \in$ UG such that $G \hookrightarrow U \mapsto N$.

▸ **Theorem 26** (EPN-reduction simulates ASG-reduction). *Let N be a proof-net. If $N \to^{*}_{ASG} G$ then there exists $N \to^{*}_{EPN} N'$ such that $G \rightarrowtail N'$.*

4.3 Quantitative unshared simulation

4.3.1 Share

▸ **Definition 27** (Copy identity). The *copy identity* labelling CID maps lifts to triples, called *labels*, of the form $x_{i:k}$ (positive) or $\overline{x}_{i:k}$ (negative), where $x \in \mathcal{V}$ a set of variable symbols, whilst $i, k \in \mathbb{N}$ are the *current* and *maximal index*. Given $U \in$ UG and μ a (tD) reduction step as in Figure 6, we set $\mathsf{CID}((v_j \mid\!\ast\rangle^n\, u_j)) = x_{j:h}$, for any $0 \le j \le h$, and for some $x \in \mathcal{V}$ not occurring in the labels of U. Labels are negated in negative lifts: if l is a positive lift such that $\mathsf{CID}(l) = x_{j:h}$ and l' is a negative residual of l w.r.t. a lift propagation rule $d\kappa$ or $r\kappa$, then $\mathsf{CID}(l') = \overline{x}_{j:h}$. In any other case, labels are preserved by reduction, in particular under copying.

▸ **Definition 28** (Sharing contexts)**.** The *sharing contexts* $\mathfrak{C}^*$ are the strings generated by the binary concatenation operator $\cdot$ over labels, and including 1 as the identity element (i.e. the empty string) and 0 as the absorbing element for concatenation. We add two equations to detect whether (or not) labels are well-bracketed in a context by neutralisation, (or nullification). For any labels $a \neq b$,

$$a \cdot \overline{a} = 1 \qquad a \cdot \overline{b} = 0. \tag{3}$$

Also, a is said *positive*, written $a > 0$, when it is empty or contains only positive labels.
A *levelled sharing context*, or simply an l-context, is a map γ from $\mathbb{N}$ to $\mathfrak{C}^*$, that is uniformly null: if for some $n \in \mathbb{N}$ we have $\gamma(n) = 0$, then $\gamma(m) = 0$ for any $m \in \mathbb{N}$. We write $(\gamma)\,|_n$ to denote the *restriction* of γ on n: if $m = n$ and $\gamma(m) \neq 0$, then $(\gamma)\,|_n(m) = 1$; otherwise $(\gamma)\,|_n(m) = \gamma(m)$. Also, we denote with $!^n a$ the *lifting* of the context a at level n. Namely, if $!^n a = \gamma$ then $\gamma(n) = a$, whilst $\gamma(m) = 1$ for any $m \neq n$. More precisely, $!^n a$ denotes $!(!^{n-1}a)$, where we set that $!1 = 1$, that $!0 = 0$, and also that $!(a \cdot b) = !a \cdot !b$. We say that γ, is *positive*, written $\gamma > 0$, if $\gamma(n) > 0$ for any n. Given a downward path $\pi : u \rightsquigarrow v$ the *l-context* of π is defined as follows.

$$\mathfrak{c}\,(()) = 1 \tag{4}$$

$$\mathfrak{c}\,(\pi :: (u,v)) = \begin{cases} \mathfrak{c}\,(\pi) \ \cdot\ !^n a & \text{if there is } l = (u \mid\!\!+\!\rangle^n\ v) \text{ s.t. } \mathsf{CID}(l) = a \\ \mathfrak{c}\,(\pi) \ \cdot\ !^n \overline{a} & \text{if there is } l = (v \mid\!\!-\!\rangle^n\ u) \text{ s.t. } \mathsf{CID}(l) = a \\ (\mathfrak{c}\,(\pi))\,|_n & \text{if there is } (u\ ?^n\ v) \\ \mathfrak{c}\,(\pi) & \text{if } u, v \text{ belong to a link of kind in } \{\overset{+}{-}\!\circ, \overset{-}{-}\!\circ, !\} \end{cases} \tag{5}$$

The *l-context of a vertex* in $U \in \mathsf{UG}$ is the context of any rooted path reaching it.

▸ **Proposition 29** (Positivity)**.** *Let π be a rooted downward path in $U \in \mathsf{UG}$. Then $\mathfrak{c}\,(\pi) > 0$.*

▸ **Proposition 30** (Path irrelevance)**.** *Let π, π' be two rooted paths in some UG ending with the vertex v. Then $\mathfrak{c}\,(\pi) = \mathfrak{c}\,(\pi')$.*

▸ **Definition 31** (Share and master)**.** A lift labelled with $x_{i:k}$ is *master* if $i = 0$, otherwise is *shared*. A context a is master if a is empty or contains only master labels $x_{0:m}$; otherwise, it is a shared context. Finally, an l-context α is master if $\alpha(n)$ is master for any $n \in \mathbb{N}$, otherwise it is shared. A vertex is shared if its l-context is so, otherwise is master; a link is shared if it has at least one shared vertex, otherwise is master. A *share component* is a non-empty, connected, and maximal (w.r.t. inclusion) sub-graph whose vertices and links are shared. The set of the share components of $U \in \mathsf{UG}$ is denoted as $\mathsf{ShC}(U)$, their union is called the *share*, written $\mathsf{Sh}(U)$.

▸ **Definition 32** (Share boundary and interior)**.** Given $U \in \mathsf{UG}$, a shared lift $l = (u \mid\!\!*\!\rangle\ v)$ is a *boundary lift*, written $l \in \mathsf{bLft}(U)$, when $u \notin \mathsf{Sh}(U) \ni v$, whilst it is an *interior lift*, written $l \in \mathsf{iLft}(U)$, when $u, v \in \mathsf{Sh}(U)$. A given $v \in \mathsf{Sh}(U)$ is *boundary*, written $v \in \mathsf{BSh}(U)$, if it is the conclusion of a boundary lift, or it is linked by a lift to a boundary vertex. If additionally v is linked to a link other than a lift, then it is *boundary-limit*, written $v \in \mathsf{BLSh}(U)$. In such a case, the *boundary lift chain* of v is the longest sequence of lifts that induces a path from v to the conclusion of a boundary lift. If $\mathsf{Sh}(U) \ni v \notin \mathsf{BSh}(U)$ and v is a ι-vertex of a lift, then v is a *pseudo-boundary vertex*, the set of which is denoted by $\mathsf{bSh}(U)$. If

Rule(s)	Proviso	$\Delta\mathsf{iSh}(\mu)$	$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu)$	$\Delta\mathsf{BShC}(\mu)$	$\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\mu)$
$(\multimap)$	$\mu \notin \mathsf{Sh}(U)$	0	9	0	9
	$\mu \in \mathsf{Sh}(U)$	$-9 + \Delta\mathsf{BShC}(\mu)$	$18 - \Delta\mathsf{BShC}(\mu)$	$[0,2]$	0
$(!)$	$\mu \notin \mathsf{Sh}(U)$	0	6	0	6
	$\mu \in \mathsf{Sh}(U)$	-6	12	$[0,1]$	0
(tD)	$\mu \notin \mathsf{Sh}(U)$	$h \times \#\mathcal{E}(B) - h$	$3h+4$	$\{0,h\}$	$h+4$
	$\mu \in \mathsf{Sh}(U)$	$h \times \#\mathcal{E}(B) - 3h - 6$	$5h+10$	0	0
$(d!)$	$l \in \mathsf{bLft}(U)$	$-3 + \Delta\mathsf{BShC}(\mu)$	$3 - \Delta\mathsf{BShC}(\mu)$	$[0,1]$	3
$(d\overset{+}{\multimap}), (d\overset{-}{\multimap}), (r\overset{-}{\multimap})$	$l \in \mathsf{bLft}(U)$	$-5 + \Delta\mathsf{BShC}(\mu)$	$5 - \Delta\mathsf{BShC}(\mu)$	$[0,2]$	5
$(d?), (r?)$	$l \in \mathsf{bLft}(U)$	$-2h - 3 + \Delta\mathsf{BShC}(\mu)$	$2h + 3 - \Delta\mathsf{BShC}(\mu)$	$[0,h+1]$	$2h+3$
$(d\kappa), (r\kappa)$	$l \notin \mathsf{bLft}(U)$	0	0	0	0
(a)	$l, l' \in \mathsf{bLft}(U)$	0	0	-1	1
	otherwise	0	0	0	0
(s)	$l, l' \notin \mathsf{bLft}(U)$, $l, l' \in \mathsf{bLft}(\mu(U))$	$-1 + \Delta\mathsf{BShC}(\mu)$	$1 - \Delta\mathsf{BShC}(\mu)$	$[0,1]$	1
	otherwise	0	0	0	0
(m)	$l \in \mathsf{bLft}(U)$	0	0	-1	1
	otherwise	0	0	0	0

Table 2 Metrics of UGR reduction: variation in the size of interior share, EPN-cost, variation in the number of boundary share components (Lemma 35); ASG-cost (Definition 37). Notations: μ is the reduction step, U is the net containing the redex, $d\kappa, r\kappa$ stands for a duplication rule; if involved: $h+1$ is the number of premisses of the ?-link, B is the box subnet, l, l' are the lifts. Also, intervals are enclosed in brackets, sets in braces.

$\mathsf{Sh}(U) \ni v \notin \mathsf{BSh}(U) \cup \mathsf{bSh}(U)$, then v is an *interior vertex*, the set of which is $\mathsf{iSh}(U)$. A share component having no interior vertices is a *boundary component*, and $\mathsf{BShC}(U)$ denotes the set of such components.

4.3.2 Proof-net cost on unshared reduction

By subtracting the variation in the size of the internal share from the classical cost of proof-net reduction, we obtain an equivalent notion of cost on unshared graphs that "translates" the work performed by global duplications into propagations of boundary lifts.

▸ **Definition 33** (EPN metrics on UG). Given $U \xrightarrow{\mu}_{\mathsf{UGR}} U'$, the *partial EPN-cost of* μ, is defined as $\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = \mathcal{C}_{\mathsf{EPN}}(\rho) - \Delta\mathsf{iSh}(\mu)$, while the *full* one is defined as $\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) + \#\mathsf{iSh}(U)$. The *boundary-share-components cost* of μ is $\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\mu) = |\Delta\mathsf{BShC}(\mu)|$.

▸ **Fact 34** (Correctness of $\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}$). *Let $N \xrightarrow{\bar{\mu}}{}^{*}_{UG} U$ and $N \xrightarrow{\bar{\rho}}{}^{*}_{UG} N'$ such that $\bar{\mu} \mapsto \bar{\rho}$. Then $\mathcal{C}_{EPN}(\bar{\rho}) = \overline{\mathcal{C}}^{EPN}_{UG}(\bar{\mu})$.*

▸ **Lemma 35** (Metrics on UGR-reduction). *Let μ be a UGR step. The possible values of $\Delta\mathsf{iSh}(\mu)$, $\mathcal{C}^{EPN}_{UG}(\mu)$ and $\Delta\mathsf{BShC}(\mu)$ are in Table 2.*

▸ Remark 36. The sum of the variations of $\#\mathsf{iSh}(\mu)$ and $\#\mathsf{BShC}(\mu)$ is constant for any step μ. Moreover, logical redexes in the share can be accounted up to twice with respect to $\mathcal{C}_{\mathsf{EPN}}$—we need to account not only for the elimination cost, but also for the unpaid instantiation cost.

4.3.3 Sharing cost on unshared reduction

By accounting a cost only to reductions that are not shared, and by distributing the cost of mux operations into lift operations, we obtain a notion of cost on unshared graphs that along a simulation is equivalent to the cost of the sharing reduction.

▸ **Definition 37** (Cost on unshared graph). Let μ be a UGR-reduction step The *ASG-cost* of μ, written $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\mu_i)$, is defined in the rightmost column of Table 2.

▸ **Lemma 38** (Master copy). *Let $N \in$ EPN, $U \in$ UG and $G \in$ ASG such that $N \xrightarrow{\bar{\sigma}}{}^*_{\mathsf{ASG}} G$ and $N \xrightarrow{\bar{\mu}}{}^*_{\mathsf{UGR}} U$, with $\bar{\sigma} \hookrightarrow \bar{\mu}$. Let $v \in V(G)$ and $V' \hookleftarrow v$. Then V' contains a unique master.*

▸ **Lemma 39** (Correctness of $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}$). *Let $N \in$ EPN, $U \in$ UG and $G \in$ ASG such that $N \xrightarrow{\bar{\sigma}}{}^*_{\mathsf{ASG}} G$ and $N \xrightarrow{\bar{\mu}}{}^*_{\mathsf{UGR}} U$, with $\bar{\sigma} \hookrightarrow \bar{\mu}$. Then $\mathcal{C}_{\mathsf{ASG}}(\bar{\sigma}) = \mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu})$.*

4.3.4 Comparison of the two unshared costs

Now we compare $\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}$ and $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}$. We find their difference to be bounded by $\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}$, and we give a quadratic bound for the latter. These two lemmas allow concluding Theorem 1.

▸ **Lemma 40.** *Let $N \in$ EPN, $U \in$ UG, such that $N \xrightarrow{\bar{\mu}}{}^*_{\mathsf{UGR}} U$. Then $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) - \mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\bar{\mu}) \leq \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})$.*

▸ **Lemma 41** (Bound to $\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}$). *For any $N \in$ EPN and any sequence $\bar{\mu}$ of UGR-reduction on N, there exists a quadratic function q such that $\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\bar{\mu}) \leq q(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu}))$.*

5 Conclusions

Two reflections and three orders of questions emerge from the study we have presented.

Discussion **1.** *A quite positive partial answer to the efficiency of sharing graphs* comes from the quadratic upper-limit to the complexity of their reductions. This is motivated by two arguments. **a.** Hypotheses of our complexity measurement were purposely extremely conservative. Indeed, we considered reduction independently from strategies, thus including those which are not lazy and does not enjoy optimality [19]. Moreover, we included also read-back rules that allow duplications of redexes, again non-optimally. **b.** The worst-case overhead of sharing graphs are ususally counterbalanced by other benefits. Laziness in the strategy of duplication, for instance, has shown speed-ups up to exponential size [3]. Locality and asynchronicity of the computational model, moreover, allow parallelisable implementation with little effort. **2.** *The cost of local duplication is legitimate.* Normalisation with sharing graphs of some ELL-typed λ-terms may cause an elementary explosion in the number of local duplication rules (mux propagations) [2]. This should not surprise, because simply-typed terms in general may require an implementation cost that is more than elementary [22]. To further clarify this point, Lemma 40 shows that duplications performed by sharing graphs have a cost that is linearly bounded by the cost of proof-net reduction.

Open questions **1.** Is the quadratic bound tight? Is there a λ-term typeable in ELL or LLL whose sharing normalisation requires indeed a cost that is quadratic with respect to proof-net reduction? **2.** Does a similar complexity upper-bound hold for the more general cases of λ-calculus and MELL? Is there also a lower bound giving theoretical evidence of performance gains? **3.** Can our bound be formulated on a proper cost-model of the λ-calculus [9]? Are sharing graphs themselves a reasonable cost model?

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A Omitted proofs

In this appendix we carefully detail all the omitted proofs of original results.

A.1 Unfolding simulation

Proof of Lemma 22. **1.** Let us consider the first statement. We proceed by induction on a ASG-reduction sequence $\bar{\sigma}$. The base case is trivial, so let $\bar{\sigma} = \bar{\sigma}'\sigma$ for some ASG-sequence $\bar{\sigma}'$ and some ASG-step σ; and let r be the redex of σ. By inductive hypothesis (IH), $\mathcal{M}'(\bar{\mu}'(N)) = \bar{\sigma}'(N)$ for some sharing morphism $\mathcal{M}'$. Consider the set of redexes R in U such that $r \hookrightarrow R$. Now, any $r' \in R$ is a redex, as implied by Definition 21, and it is disjoint to any $r'' \in R$, by orthogonality of the definition of UG-reduction. So let $\bar{\mu}$ be a reduction sequence reducing all and only $r' \in R$. It is now easy to define $\mathcal{M}$ from $\mathcal{M}'$, so that it maps any residual of r' into the residual of r. Hence, $\mathcal{M}(\bar{\mu}(\bar{\mu}'(N))) = \sigma(\bar{\sigma}'(N))$, and we conclude.

2. The second statement can be proved almost identically to first one. The only difference worth to mention is the fact that the set of UG-redexes simulating a (m)-step overlaps, but this poses no problem, since they enjoy pair-wise confluence. ◂

Proof of Lemma 24. By a trivial induction on the length of the UG reduction sequence. ◂

Proof of Theorem 26. Immediate from from Lemma 22 and Lemma 24. ◂

A.2 Statics of downward paths

▸ **Definition 42** (Variables)**.** Given a $N \in$ EPN, we denote by $\mathsf{BVar}(N)$ the set of the vertices which are connected to the first premiss of a $\overset{\pm}{\multimap}$-link; and by $\mathsf{FVar}(N)$ the set of the inputs of N; then, $\mathsf{FBVar}(N) = \mathsf{BVar}(N) \cup \mathsf{FVar}(N)$. The same notions are naturally extended to ASG and UG.

▸ **Lemma 43.** *$\rightsquigarrow$ is a partial order on vertices of any proof-net.*

Proof of Lemma 43. Reflexivity and transitivity of $\rightsquigarrow$ are trivially verified against the definition of the relation itself. What about antisymmetry, which is equivalent to the absence of cycles? Downward paths are a restriction of switching paths[1], which allows only one of the two possible switchings on $\multimap$-links (the principal one) and which imposes the downward direction. But switching paths are necessarily acyclic, thus so are downward paths. ◂

▸ **Lemma 44** (Connectivity)**.** *For any vertex v in a proof-net G:*

1. *$v \leftsquigarrow u$ with u root of G;*
2. *$v \rightsquigarrow u'$ with $u' \in \mathsf{FBVar}(G)$;*
3. *if $v \rightsquigarrow u$ and there is a link $(u, w \overset{\pm}{\multimap} z)$, then $v \sim w$;*
4. *if v is the root of G or the premiss of a $!$-link, and u is the premiss of a $?$-link h, with $\ell(u) > \ell(v)$ and $v \rightsquigarrow u$, then for any w premiss of h there exists $\pi : v \rightsquigarrow z \rightsquigarrow w$, such that z is the premiss of principal door of the box of w.*

Proof. By a simple induction on the definition of EPN. ◂

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▸ **Lemma 45.** $\rightsquigarrow$ *is a partial order on vertices of unshared graphs.*

Proof. Immediate from Lemma 43. ◂

▸ **Lemma 46** (Crossing links)**.** *Let π be a downward path in a $U \in$ UG and $l \in L(U)$. If is not a $\overset{\pm}{-\!\circ}$-link, then π contains at most 2 vertices of l; otherwise π contains at most 3 vertices, and in such case one is the lowermost extremum of π.*

Proof. We separately consider the two disjointed proposition of the claim.

1. Suppose l is not $\overset{\pm}{-\!\circ}$-link. Then let u, v be two distinct vertices of π, respectively being an o-vertex and an ι-vertex of the link l. Then (u, v) is by definition a downward path over l. Now clearly u must precede v in π, since otherwise, as per antisymmetry property of $\rightsquigarrow$ (Lemma 45), we would contradict our distinctness hypothesis. Moreover, u, v have to be consecutive in π, because the existence of $\pi' \neq (u, v)$ would imply that (i) there exist two links l, l' both insisting on u with their ι-vertex, and similarly that (ii) there exist two links l, l'' both insisting on v with their o-vertex. Indeed, both these facts would be absurd, since at most one ι- and one o-vertex connection are allowed by definitions of unshared graphs. So let $\pi = \pi' :: (u, v) :: \pi''$, and by acyclicity we conclude that there cannot be other be other occurrences of u or v within π' or π''.
2. Suppose otherwise that l is a $\overset{\pm}{-\!\circ}$. In particular, let $l = (w, v \overset{\pm}{-\!\circ} u)$. Exactly as in the previous case, we have that u, v must be appear consecutively in such order within π, so let $\pi = \pi' :: (u, v) :: \pi''$. Moreover, u, v cannot occur elsewhere in π' nor π''. Now, by definition of downward path there exists no w' such that $w \rightsquigarrow w'$, therefore it must be the last and possibly unique vertex of π''. Hence the claim.

◂

▸ **Lemma 47** (Connectivity)**.** *For any vertex in $U \in$ UG,*

1. *$v \leftsquigarrow u$ with u root of U,*
2. *$v \rightsquigarrow u'$ with u'FBVar(U),*
3. *if $v \rightsquigarrow u$ and there is a link $(u, w \overset{\pm}{-\!\circ} z)$, then $v \sim w$.*
4. *if v is the root of U or the premiss of a $!$-link, and u is the premiss of a $?$-link h, with $\ell(u) > \ell(v)$ and $v \rightsquigarrow u$, then for any w premiss of h there exists $\pi : v \rightsquigarrow z \rightsquigarrow w$, such that z is the premiss of principal door of the box of w.*

Proof. Immediate from Lemma 44. ◂

A.3 Dynamics of downward paths in unshared graphs

▸ **Definition 48** (Redex crossing)**.** Given a path π and a redex R in a structure in EPN, UG, ASG, any sub-path $\pi' \subseteq \pi$ is a *crossing of* R if every vertex of π' belongs to the vertices of R. If there is no path π'' being a crossing of R and such that $\pi' \subsetneq \pi'' \subseteq \pi$, then π' is a *maximal crossing of* R. Moreover, if all crossings of π with respect to R are maximal, then π is *long enough for* R.

▸ **Definition 49** (Unshared reduction of downward maximal crossing)**.** Let χ be a downward path in $U \in$ UG being a maximal crossing for a UG-redex R, and let ρ be its reduction. The *reduction of* χ *with respect to* ρ is the set of downward path defined as follows.

1. If R is a $(\multimap)$-step, let it be as in Figure 4a. Then:

$$\rho((v_2, w, u_2)) = \{(u_2 = v_2)\} \tag{6}$$
$$\rho((u_1)) = \{(u_1 = v_1)\} \tag{7}$$
$$\rho((v_2, v_1)) = \{(u_2 = v_2) :: \gamma :: (u_1 = v_1) \mid \gamma : u_2 \rightsquigarrow u_1\} \tag{8}$$

The rightmost side of (8) would be the empty set if $u_2 \not\rightsquigarrow u_1$, but it never the case, as per Item 3.
2. If R is a $(!)$-step, let it be as in Figure 5a Then:

$$\rho((v_0, w, u)) = \{(v_0 = u)\} \tag{9}$$

3. If R is a $(d\overset{+}{\multimap})$-step, let it be as in Figure 5c. Then:

$$\rho((v_0, w, u_2)) = \{(v_0, z_{2_0}, u_2)\} \tag{10}$$
$$\rho((u_1)) = \{(u_1)\} \tag{11}$$

4. If R is a $(d\overset{-}{\multimap})$-step, let it be as in Figure 5d. Then:

$$\rho((u_2, w, v_0)) = \{(u_2, z_{2_0}, v_0)\} \tag{12}$$
$$\rho((u_2, u_1)) = \{(u_2, z_{2_0}, u_{2_0}, u_1)\} \tag{13}$$

5. If R is a $(d!)$-step, let it be as in Figure 5e. Then:

$$\rho((v_0, w, u)) = \{(v_0, z_0, u)\} \tag{14}$$

6. If R is a $(d?)$-step, let it be as in Figure 5f. Then:

$$\rho((u_i, w, v_0)) = \{(u_i, z_{0_i}, v_0)\} \tag{15}$$

where $0 \leq i \leq h$.
7. If R is a (a)-step, let it be as in Figure 5b. Then:

$$\rho((u_0, w, v_0)) = \{(u_0 = v_0)\} \tag{16}$$

8. If R is a (s)-step, let it be as in Figure 5b. Then:

$$\rho((u_0, w, v_0)) = \{(u_0, z_{0_0}, v_0)\} \tag{17}$$

9. Otherwise R is a (tD)-step. Let it be as in Figure 6. Then:

$$\rho((v_i, z, w) :: \gamma :: (u'_j, u_j)) = \{(v_i, w_i) :: \gamma_i :: (u'_{j_i}, u_j)\} \tag{18}$$

where:
- $0 \leq i \leq h$, and $1 \leq j \leq k$;
- $u'_j \in U_j$ is a premiss of the ?-link having conclusion in u_j;
- $\gamma : w \rightsquigarrow u'_j$ is a path in the subnet b;
- γ_i is the i-th copy of γ in the subnet b_i.

▸ **Definition 50** (Unshared read-back reduction of downward maximal crossing)**.** Let χ be a downward path in a $U \in \mathsf{UG}$ being a maximal crossing for a UR-redex R, and let ρ be its reduction. The *reduction of* π *with respect to* ρ is the set of downward path defined as follows.

1. If R is a $(r\multimap)$-step, let it be as in Figure 5g. Then:

$$\rho((v_0, u_2, u_0)) = \{(v_0, z_{0_0}, u_0)\} \tag{19}$$
$$\rho((v_0, u_2, u_1)) = \{(v_0, z_{1_0}, u_1)\} \tag{20}$$

2. If R is a $(r?)$-step, let it be as in Figure 5h. Then:

$$\rho((v_0, u_i, w)) = \{(v_0, y_0, w)\} \tag{21}$$
$$\rho((u_j, w)) = \{(u_j, z_{j_0}, z_{0_0}, w)\} \tag{22}$$

where $j \neq i$.

3. If R is a (m)-step, let it be as in Figure 5h. Then:

$$\rho((v_0, u_i, w)) = \{(v_0, w)\} \tag{23}$$
$$\rho((u_j, w)) = \{(u_j, w)\} \tag{24}$$

where $j \neq i$.

▸ **Definition 51** (Reduction of downward paths)**.** Let π be a downward path in $U \in \mathsf{UG}$ and let R be a UG- or a UR-redex such that π is long enough for R. If ρ is the reduction step of R, then the *reduction of* π *with respect to* ρ is a function from π to a set $\rho(\pi)$ of downward paths in $\rho(U)$. Let

$$\pi = \pi_0 :: \chi_1 :: \pi_1 :: \ldots :: \chi_n :: \pi_n \tag{25}$$

where for any $0 \leq i \leq n$ the sub-path χ_i is a maximal crossing of a redex R. Then:

$$\rho(\pi) = \{\pi_0 :: \chi'_1 :: \pi_1 :: \ldots :: \chi'_n :: \pi_n \mid \chi'_i \in \rho(\chi_i)\} \tag{26}$$

where the reduction of a maximal crossing of R is defined in Definition 49 and 50.

A.3.1 Sharing contexts

▸ **Definition 52** (Stable form)**.** Given a context a, we denote with $\overline{a}$ the context obtained by reversing a and inverting, in an involutive fashion, the positivity of each of its labels. Namely, given any labels $x_{i:k}, \overline{x}_{i:k}, b$,

$$\overline{x_{i:k}} = \overline{x}_{i:k} \tag{27}$$
$$\overline{\overline{x}_{i:k}} = x_{i:k} \tag{28}$$
$$\overline{\overline{a}} = a \tag{29}$$
$$\overline{a \cdot b} = b \cdot a \tag{30}$$

Moreover, we call a negative, written $a < 0$ if $\overline{a}$ is positive. Observe that 1 is both positive and negative, whilst 0 is neither. A *stable form* of a context c is a context $a \cdot b = c$ such that $a < 0$ and $b > 0$.

▸ **Lemma 53** (Stability or nullity)**.** *Any context is equal either to* 0 *or to a unique stable form.*

Proof sketch. By orienting $\mathfrak{C}^*$ equations (see Equation (3)) from left to right we obtain a very simple rewriting system. Trivially, we observe that it is terminating, since the length of a context is strictly decreasing, and locally confluent, because of orthogonality. ◂

▸ **Lemma 54** (Neutrality). *If $a, b \in \mathfrak{C}^*$ such that $a \cdot b = a \neq 0$, then there exists $c, d \in \mathfrak{C}^*$ such that $a = d \cdot c$ and $b = \overline{c} \cdot c$, with c positive.*

Proof. We first observe that $b \neq 0$. Indeed, if we suppose otherwise we would obtain that $a \cdot b = 0$. But $a \cdot b = a$, therefore we would have $a = 0$ contradicting our hypothesis. So a, b are not null, hence they have a stable form.

$$a = d' \cdot d \quad \text{(stable form)} \tag{31}$$
$$b = c' \cdot c \quad \text{(stable form)} \tag{32}$$
$$a \cdot b = d' \cdot d \cdot c' \cdot c \quad (31), (32) \tag{33}$$

Now, $d \cdot c'$ cannot be null, so they have a stable form.

$$d' \cdot c = e' \cdot e \quad \text{(stable form)} \tag{34}$$
$$a \cdot b = d' \cdot e' \cdot e \cdot c \quad (34), (33) \tag{35}$$
$$d' \cdot e' \cdot e \cdot c = d' \cdot d \quad \text{(hypothesis)}, (35) \tag{36}$$
$$e' \cdot e \cdot c = d \quad (36) \tag{37}$$

Observe that by definition $d > 0$, while $e' < 0$. Therefore $e' > 0$, i.e.

$$e' = 1 \tag{38}$$
$$e \cdot c = d \quad (38), (37) \tag{39}$$
$$a = d' \cdot e \cdot c \quad (31), (39) \tag{40}$$

which is our first claim. Moreover,

$$d' \cdot e \cdot c = d' \cdot e \cdot c \cdot c' \cdot c \quad \text{(hypothesis)}, (32), (40) \tag{41}$$
$$1 = c \cdot c' \quad (41) \tag{42}$$
$$c' = \overline{c} \quad (42) \tag{43}$$
$$b = \overline{c} \cdot c \quad (32), (43) \tag{44}$$

that is the second claim. ◂

A.3.2 Positivity and syntactic compatibility of sharing l-contexts

We now prove two fundamental properties of contexts. The first is that positive and negative lifts are placed along downward paths as matching parentheses, which algebraically means that contexts are always positive. The second is that any reduction different from $d \overset{+}{-}\!\circ$-rule does not affect contexts of downward paths.

▸ **Definition 55** (Generalised equality and positivity). Let γ, γ' be two l-contexts. We shall write $\gamma =_n \gamma'$ to mean that $\gamma(m) = \gamma(m)$ for any $m \geq n$, while $\gamma =^n \gamma'$ dually means that $\gamma(m) = \gamma(m)$ for any $m < n$. Moreover, $\gamma >_n 0$ denotes the fact that $\gamma(m) > 0$ for any $m \geq n$.

▸ **Proposition 56** (Strong positivity). *Let π be a downward path in a $U \in UG$ having maximum in v, being the root of U or the premiss of a !-link. Then $\mathfrak{c}(\pi) >_{\ell(v)} 0$.*

The proof requires four additional properties.

▸ **Lemma 57** (Long Invariance). *If π is a downward path in a $U \in$ UG long enough for the redex of a reduction step ρ, then $\mathfrak{c}(\pi) = \mathfrak{c}(\rho(\pi))$ for any $\pi' \in \rho(\pi)$.*

▸ **Lemma 58** (Lambda-compatibility). *Let $\pi : u \rightsquigarrow w$ and $\pi' : w \rightsquigarrow v$ in $U \in$ UG containing $(v, w \overset{+}{\multimap} u')$. If u is the root of U or the premiss of a $!$-link, then $\mathfrak{c}(\pi) =_{\ell(u)} \mathfrak{c}(\pi :: \pi')$.*

▸ **Lemma 59** (Box compatibility). *Let $\pi : u \rightsquigarrow v$ and $\pi' : u \rightsquigarrow v'$ in $U \in$ UG such that: u is the root of U or a premiss of a $!$-link, and v, v' are premisses of the same $?$-link. Then $\mathfrak{c}(\pi) =^{\ell(v)}_{\ell(u)} \mathfrak{c}(\pi')$.*

▸ **Lemma 60** (Unary contraction). *Let $\pi : u \rightsquigarrow v$ be a downward path in a $U \in$ UG, where u is a $!$-premiss, v is a $?$-premiss. If u, v belong to a box B, then $\mathfrak{c}(v)(\ell(v)) = 1$.*

The four lemmas and the positivity proposition are so tightly related one with the other that their proof will require a mutually recursive approach: to prove a statement we will use another as inductive hypothesis.

Proof of Lemma 57 (Long Invariance). Let R be the redex of ρ. Since π is long enough for R, let $\pi = \pi_0 :: \chi_1 :: \pi_1 :: \ldots :: \chi_n :: \pi_n$, where χ_c is are maximal crossing of R, for any $1 \leq c \leq n$. Recall from Lemma 46 that a link l could be crossed by a downward path at most: twice, if l is not a $\overset{+}{\multimap}$-link, three times, if it is. But a redex contains two connected links, therefore, if R does not contain a $\overset{+}{\multimap}$-link then $n = 1$, otherwise $n \leq 2$. Now, if $n = 0$ then $\rho(\pi) = \{\pi\}$, and both claims trivially hold. So we assume otherwise that $n > 0$, and proceed by a case analysis on the kind of rule employed by ρ'.

1. Rule $(\multimap)$ and $n \leq 2$. Let R be as in Figure 4a Recall from Definition 49 that χ can be of three kinds, among which (u_i) is quite peculiar, since by Lemma 46, it does not allow right concatenation, and is the only one that can succeed another maximal crossing of the same redex R, namely (v_2, w, u_2). For these reasons, and by our hypotheses on π, we can simplify our analysis to the following four cases.
   a. Suppose that π crosses (v_2, w, u_2). Then by (6), we have $\rho(\chi) = \{(v_2 = u_2)\}$, and $\rho(\pi) = \{\pi_0 :: (v_2 = u_2) :: \pi_1\}$. So let $\pi' = \pi_0 :: \pi_1$ be the only element of $\rho(\pi)$. Now, by definition of context assignment, $\mathfrak{c}(\chi) = \mathfrak{c}(\rho(\chi)) = 1$. Therefore, $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$.
   b. Assume that π crosses (v_2, v_1). This is one of the most interesting cases of this proof. By (8) and (26), we have

$$\rho(\chi) = \{(u_2 = v_2) :: \gamma :: (u_1 = v_1) \mid \gamma : u_2 \rightsquigarrow u_1\} \tag{45}$$

$$= \{\gamma \mid \gamma : u_2 \rightsquigarrow u_1\}, \tag{46}$$

$$\rho(\pi) = \rho(\pi_0 :: \chi :: \pi_1) \tag{47}$$

$$= \{\pi_0 :: \gamma :: \pi_1 \mid \gamma : u_2 \rightsquigarrow u_1\}. \tag{48}$$

   Now consider $\pi' = \pi_0 :: \gamma :: \pi_1 \in \rho(\pi)$, for some $\gamma : u_2 \rightsquigarrow u_1$. By definition of context,

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0 :: \gamma :: \pi_1) \tag{49}$$

$$= \mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\gamma) \cdot \mathfrak{c}(\pi_1). \tag{50}$$

   Now we can safely apply Lemma 58 on γ and obtain invariance.

$$= \mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\pi_1) \tag{51}$$

$$= \mathfrak{c}(\pi_0) \cdot 1 \cdot \mathfrak{c}(\pi_1) \tag{52}$$

$$= \mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\chi) \cdot \mathfrak{c}(\pi_1) \tag{53}$$

$$= \mathfrak{c}(\pi) \tag{54}$$

c. If π crosses (u_1), we immediately observe that $\rho(\pi) = \{\pi\}$, which trivially means that $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$, for any $\pi' \in \rho(\pi)$.

d. If π crosses R first in (v_2, w, u_2) and then in (u_1), exactly as we noticed in case 1a, we have a unique $\pi' \in \rho(\pi)$, that is $\pi_0 :: \pi_1$. Thus, $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$.

2. Rule (!) and $n = 1$. Let R as in the leftmost part of Figure 5a, let m be the depth of the !-link, and let also $\chi_1 = (v_0, w, u)$, that is the only possible maximal crossing of R. By definition, its context is as follows:

$$\mathfrak{c}(\pi) = (\mathfrak{c}(\pi_0))|_m \cdot \mathfrak{c}(\pi_1). \tag{55}$$

By definition of reduction, as per (9):

$$\begin{aligned} \rho(\pi) &= \rho(\pi_0 :: (v_0, w, u) :: \pi_1) && (56)\\ &= \{\pi_0 :: (v_0 = u) :: \pi_1\} && (57)\\ &= \{\pi_0 :: \pi_1\}. && (58) \end{aligned}$$

Let $\pi' \in \rho(\pi)$. Its context is

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\pi_1). \tag{59}$$

So, let $m' \geq \ell(v)$.

a. If $m' \neq m$, then $(\mathfrak{c}(\pi_0))|_m(m') = \mathfrak{c}(\pi_0)(m')$, as by Definition 28 of restriction operator. Hence, $\mathfrak{c}(\pi')(m') = \mathfrak{c}(\pi)(m')$. Since by inductive hypothesis (IH) we have that $\mathfrak{c}(\pi)(m') > 0$, we also have $\mathfrak{c}(\pi')(m') > 0$.

b. Otherwise, $m' = m$. Observe that, by definition of proof-nets, v_0 belong to a box B, so let $\pi_0 = \eta :: \beta$, where β is the maximal suffix of π_0, whose vertices are in B. Hence by definition:

$$\begin{aligned} \mathfrak{c}(\pi)(m) &= \mathfrak{c}(\eta)(m) \cdot (\mathfrak{c}(\beta))|_m(m) \cdot \mathfrak{c}(\pi_1)(m)\\ &= \mathfrak{c}(\eta)(m) \cdot 1 \cdot \mathfrak{c}(\pi_1)(m). && (60) \end{aligned}$$

Now, by construction, the maximum of β is an !-premiss of B, and v_0 is the only ?-premiss of B, therefore we can apply Lemma 60, to obtain that $\mathfrak{c}(\beta)(m) = 1$.

$$\begin{aligned} \mathfrak{c}(\pi')(m) &= \mathfrak{c}(\eta)(m) \cdot \mathfrak{c}(\beta)(m) \cdot \mathfrak{c}(\pi_1)(m)\\ \mathfrak{c}(\pi')(m) &= \mathfrak{c}(\eta)(m) \cdot 1 \cdot \mathfrak{c}(\pi_1)(m). && (61) \end{aligned}$$

Therefore $\mathfrak{c}(\pi)(m) = \mathfrak{c}(\pi)(m)$.

3. Rule (tD) and $n = 1$. Let R as in Figure 6, and if l' is the ?-link in R, then let $m = \ell(l')$ and $a = \mathsf{CID}(l')$. Observe that by definition, χ_1 is in the form $(v_j, z, w) :: \gamma :: (U_{j'}, u_{j'})$, where $0 \leq j \leq h$ with $h > 0$, $1 \leq j' \leq k$, and γ is a downward path in the box b. Remember that by definition of reduction, and in particular by (18),

$$\begin{aligned} \rho(\pi) &= \rho(\pi_0 :: (v_j, z, w) :: \gamma :: (U_{j'}, u_{j'}) :: \pi_1)\\ &= \left\{\pi_0 :: (v_j, w_i) :: \gamma_j :: (U_{j'_j}, u_j) :: \pi_1\right\}. && (62) \end{aligned}$$

Its context is by definition:

$$\begin{aligned} \mathfrak{c}(\pi) &= \mathfrak{c}(\pi_0 :: (v_j, z, w) :: \gamma :: (U_{j'}, u_{j'}) :: \pi_1)\\ &= (\mathfrak{c}(\pi_0))|_m \cdot (\mathfrak{c}(\gamma))|_m \cdot \mathfrak{c}(\pi_1) && (63) \end{aligned}$$

Let π' is the unique element of $\rho(\pi)$, and observe it crosses the j-th lift introduced by the reduction, say l''. By definition of reduction and that of variable occurrences, $\ell(l'') = \ell(l') = m$ and $\mathsf{CID}(l'') = \mathsf{CID}(l') = a$. Therefore,

$$\begin{aligned}\mathfrak{c}(\pi') &= \mathfrak{c}\Big(\pi_0 :: (v_j, w_j) :: \gamma_j :: (U_{j'_j}, u_j) :: \pi_1\Big) \\ &= (\mathfrak{c}(\pi_0) \cdot !^m a \cdot \mathfrak{c}(\gamma))|_m \cdot \mathfrak{c}(\pi_1). \end{aligned} \tag{64}$$

Let $m' \geq \ell(v)$ and consider its equality with respect to m.

a. If $m' \neq m$, we easily conclude. Indeed, by definitions of restriction and lifting we obtain the claim from (63) and (64).

$$\begin{aligned}\mathfrak{c}(\pi)(m') &= (\mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\gamma) \cdot \mathfrak{c}(\pi_1))(m') \\ &= \mathfrak{c}(\pi')(m'). \end{aligned} \tag{65}$$

b. Otherwise, $m' = m$. Consider again (63) and (64), and recall that, applying Proposition 56 of positivity as IH, $\mathfrak{c}(\pi_0) >_{\ell(v)} 0$ and $\mathfrak{c}(\gamma) >_m 0$. In particular, these imply that respectively $\mathfrak{c}(\pi_0)(m) > 0$ and $\mathfrak{c}(\gamma)(m) > 0$. Now, by definition of restriction and lifting operators, we easily rewrite (63) into:

$$\mathfrak{c}(\pi)(m) = 1 \cdot 1 \cdot (\mathfrak{c}(\pi_1))(m) \tag{66}$$

$$= \mathfrak{c}(\pi_1)(m). \tag{67}$$

Now, in (64) we can exploit again positivity as IH to obtain $a \cdot \mathfrak{c}(\gamma)(m) > 0$. Therefore we conclude.

$$\mathfrak{c}(\pi')(m) = 1 \cdot (\mathfrak{c}(\pi_1))(m) \tag{68}$$

$$= \mathfrak{c}(\pi)(m). \tag{69}$$

4. Rule $(d!)$ and $n = 1$. Assume R is as in the left of Figure 5e, where $\chi_1 = (v_0, w, u)$. Also, if l is the ⟨+⟩-link in R, let n be its level, and let $a = \mathsf{CID}(l)$ for some $a \in \mathfrak{C}^*$. By definition of context assignment,

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0 :: (v_0, w, u) :: \pi_1) \tag{70}$$

$$= \mathfrak{c}(\pi_0) \cdot !^n a \cdot 1 \cdot \mathfrak{c}(\pi_1) \tag{71}$$

By definition of reduction, and in particular by (14):

$$\rho(\pi) = \rho(\pi_0 :: (v_0, w, u) :: \pi_1) \tag{72}$$

$$= \{\pi_0 :: (v_0, z_0, u) :: \pi_0\} \tag{73}$$

So let π' be the unique path in $\rho(\pi)$ and consider its context assignment.

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0 :: (v_0, z_0, u) :: \pi_1) \tag{74}$$

$$= \mathfrak{c}(\pi_0) \cdot 1 \cdot !^n a \cdot \mathfrak{c}(\pi_1) \tag{75}$$

$$= \mathfrak{c}(\pi) \tag{76}$$

5. Rule $(d?)$ and $n = 1$. Let R as in the right of Figure 5f, and let $\chi_1 = (u_i, w, v_0)$, for some $0 \leq i \leq h$. Also, if l is the ⟨+⟩-link in R, let n be its level, and let $a = \mathsf{CID}(l)$ for some $a \in \mathfrak{C}^*$. Finally, let $m > n$ be the level of the ?-link. By definition of context assignment,

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0 :: (u_i, w, v_0) :: \pi_1) \tag{77}$$

$$= (\mathfrak{c}(\pi_0))|_m \cdot !^n a \cdot \mathfrak{c}(\pi_1) \tag{78}$$

By definition of reduction, and in particular by (15):

$$\rho(\pi) = \rho(\pi_0 :: (u_i, w, v_0) :: \pi_1) \tag{79}$$
$$= \{\pi_0 :: (u_i, z_{0_i}, v_0) :: \pi_0\} \tag{80}$$

So let π' be the unique path in $\rho(\pi)$ and consider its context assignment, and observe it is equivalent by definition of the restriction operator.

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0 :: (u_i, z_{0_i}, v_0) :: \pi_1) \tag{81}$$
$$= (\mathfrak{c}(\pi_0) \cdot !^n a)|_m \cdot \mathfrak{c}(\pi_1) \tag{82}$$
$$= \mathfrak{c}(\pi) \tag{83}$$

6. Rule $(d\overset{+}{\multimap})$ and $1 \leq n \leq 2$. Let R as in Figure 5c, and if h is the lift within R, then let $a = \mathsf{CID}(h)$ and let m be its level.

a. Suppose that π contains only the maximal crossing (v_0, w, u_2). By definition of context assignment we have that

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0 :: (v_0, w, u_2) :: \pi_1)) \tag{84}$$
$$= \mathfrak{c}(\pi_0 :: (v_0, w) :: (w, u_2) :: \pi_1) \tag{85}$$
$$= \mathfrak{c}(\pi_0) \cdot !^m a \cdot 1 \cdot \mathfrak{c}(\pi_1)\,. \tag{86}$$

Now, by definition of downward reduction, in particular (10),

$$\rho(\pi) = \rho(\pi_0 :: (v_0, w, u_2) :: \pi_1) \tag{87}$$
$$= \{\pi_0 :: (v_0, z_{2_0}, u_2) :: \pi_1\}, \tag{88}$$

So if we consider the context of $\pi' \in \rho(\pi)$, we easily obtain that is equal to that of π.

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0 :: (v_0, z_{2_0}, u_2) :: \pi_1) \tag{89}$$
$$= \mathfrak{c}(\pi_0) \cdot \mathfrak{c}((v_0, z_{2_0}) \cdot \mathfrak{c}((z_{2_0}, u_2)) \cdot \mathfrak{c}(\pi_1) \tag{90}$$
$$= \mathfrak{c}(\pi_0) \cdot 1 \cdot !^m a \cdot \mathfrak{c}(\pi_1) \tag{91}$$
$$= \mathfrak{c}(\pi)\,. \tag{92}$$

b. Assume that π crosses R only in (u_1). By definition of context assignment, and of reduction we have:

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0 :: (u_1)) = \mathfrak{c}(\pi_0)\,, \tag{93}$$
$$\rho(\pi) = \rho(\pi_0 :: (u_1)) = \{\pi_0 :: (u_1)\}. \tag{94}$$

So, we immediately conclude: let $\pi' \in \rho(\pi)$ and consider its context.

$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0 :: (u_1)) = \mathfrak{c}(\pi)\,. \tag{95}$$

c. Otherwise π crosses R twice, and it must be the case that $\chi_1 = (v_0, w, u_2)$, while $\chi_2 = (u_1)$. From previous cases 6a and 6b we have: $\mathfrak{c}(\chi_1) = \mathfrak{c}(\chi_1')$ for any $\chi_1' \in \rho(\chi_1)$, and $\mathfrak{c}(\chi_2) = \mathfrak{c}(\chi_2')$ for any $\chi_2' \in \rho(\chi_2)$. Hence the claim.

7. Rule $(d\overset{-}{\multimap})$ and $n = 1$. Let R as in the rightmost part of Figure 5d, and if h is the lift within R, then let $a = \mathsf{CID}(h)$ and $m = \ell(h)$. We distinguish two cases, according to the two possible maximal crossing of R.

a. If $\chi_1 = (u_2, w, v_0)$, then by definition of reduction, and in particular by (12),

$$\rho(\pi) = \rho(\pi_0 :: (u_2, w, v_0) :: \pi_1) \tag{96}$$
$$= \{\pi_0 :: (u_2, z_{2_0}, v_1) :: \pi_1\} \tag{97}$$

Let $\pi' \in \rho(\pi)$. By simply inspecting the definition, we immediately verify that the context is unvaried.

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0) \cdot 1 \cdot !^m\overline{a} \cdot \mathfrak{c}(\pi_1) \tag{98}$$
$$= \mathfrak{c}(\pi_0) \cdot !^m\overline{a} \cdot \mathfrak{c}(\pi_1) \tag{99}$$
$$= \mathfrak{c}(\pi_0) \cdot !^m\overline{a} \cdot 1 \cdot \mathfrak{c}(\pi_1) \tag{100}$$
$$= \mathfrak{c}(\pi') \tag{101}$$

b. Otherwise $\chi_1 = (u_2, u_1)$. Its reduction is by (13),

$$\rho(\pi) = \rho(\pi_0 :: (u_2, u_1) :: \pi_1) \tag{102}$$
$$= \{\pi_0 :: (u_2, z_{2_0}, u_{2_0}, u_1) :: \pi_1\}. \tag{103}$$

While the context of π and that of $\pi' \in \rho(\pi)$ are:

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0) \cdot 1 \cdot \mathfrak{c}(\pi_1) \tag{104}$$
$$\mathfrak{c}(\pi') = \mathfrak{c}(\pi_0) \cdot !^m\overline{a} \cdot !^n a \cdot \mathfrak{c}(\pi_1). \tag{105}$$

Now, let $m' \geq l$ and consider its equality with respect to m.

i. If $m' \neq m$, then $\mathfrak{c}(\pi)(m) = \mathfrak{c}(\pi')(m)$.

ii. Otherwise, $m' = m$. By IH we have that $\mathfrak{c}(\pi_0 :: (u_2, u_1)) >_{\ell(v)} 0$, which means that in particular $\mathfrak{c}(\pi_0)(m) \cdot \overline{a} > 0$. Therefore, it must be the case that $\mathfrak{c}(\pi_0)(m) = b \cdot a$, for some $b \in \mathfrak{C}^*$. Substituting it in (104) and (105), we obtain

$$\mathfrak{c}(\pi)(m) = b \cdot a \cdot \mathfrak{c}(\pi_1)(m) \tag{106}$$
$$\mathfrak{c}(\pi')(m) = b \cdot a \cdot \overline{a} \cdot a \cdot \mathfrak{c}(\pi_1)(m) \tag{107}$$
$$= b \cdot a \cdot \mathfrak{c}(\pi_1)(m). \tag{108}$$

8. Rule $(r?)$ and $n = 1$. Let R as in Figure 5h, where h is the ⟬+⟭-link and h' is the ?-link involved R. Let $m = \ell(h)$, $a = \mathsf{CID}(h)$. We have two kinds of downward crossings, which we analyse separately.

a. Assume first $\chi_1 = (v_0, u_i, w)$. By definition of reduction and in particular by (21),

$$\rho(\pi) = \rho(\pi_0 :: (v_0, u_i, w) :: \pi_1) \tag{109}$$
$$= \{\pi_0 :: (v_0, y_0, w) :: \pi_1\}. \tag{110}$$

Let π' be the unique path in $\rho(\pi)$ and inspect the definition of context assignment. We immediately verify it is unvaried.

$$\mathfrak{c}(\pi) = \mathfrak{c}(\pi_0) \cdot !^m\overline{a} \cdot 1 \cdot \mathfrak{c}(\pi_1) \tag{111}$$
$$= \mathfrak{c}(\pi_0) \cdot 1 \cdot !^m\overline{a} \cdot \mathfrak{c}(\pi_1) \tag{112}$$
$$= \mathfrak{c}(\pi') \tag{113}$$

b. Otherwise, hypothesise that $\chi_1 = (u_j, w)$ where $i \neq j$. By definition and in particular by (22):

$$\begin{aligned}\rho(\pi) &= \rho(\pi_0 :: (u_j, w) :: \pi_1) &(114)\\ &= \{\pi_0 :: (u_j, z_{j_0}, z_{0_0}, w) :: \pi_1\}. &(115)\end{aligned}$$

Now, let $m' \geq \ell(v)$ and consider its equality with respect to m.

i. If $m' \neq m$, then by definition of the lifting operator we immediately obtain that the context is invariant and that IH is consequently preserved.

$$\begin{aligned}\mathfrak{c}(\pi)(m') &= (\mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\pi_1))(m') &(116)\\ &= \mathfrak{c}(\pi')(m') &(117)\end{aligned}$$

ii. Otherwise, we have that:

$$\begin{aligned}\mathfrak{c}(\pi)(m) &= \big(\mathfrak{c}(\pi_0) \cdot \mathfrak{c}(\pi_1)\big)(m), &(118)\\ \mathfrak{c}(\pi')(m) &= \big(\mathfrak{c}(\pi_0) \cdot !^m\overline{a} \cdot !^m a \cdot \mathfrak{c}(\pi_1)\big)(m). &(119)\end{aligned}$$

Let π_0' be a downward path whose first vertex is the same as that of π_0, and whose last vertex is the premiss u_i of h. Its existence is guaranteed by Item 4. As observed in previous case 8a, let $\mathfrak{c}(\pi_0')(m) = \alpha \cdot \overline{a}$, for some $\alpha \in \mathfrak{C}^*$. By Lemma 59, we know that $\mathfrak{c}(\pi_0) =_{\ell(v)}^{\ell(h')} \mathfrak{c}(\pi_0')$, therefore we obtain the claim:

$$\begin{aligned}\mathfrak{c}(\pi')(m) &= \alpha \cdot a \cdot \overline{a} \cdot a \cdot \mathfrak{c}(\pi_1) &(120)\\ &= \alpha \cdot a \cdot \mathfrak{c}(\pi_1)(m) &(121)\\ &= \mathfrak{c}(\pi)(m). &(122)\end{aligned}$$

9. Rule $(r\bar{\multimap})$. Omitted: an inspection of definitions shows that for each of the two possible downward crossing of R the situation is identical to what is described in case 4 (rule $(d!)$).

10. Rule (s) and $n = 1$. Let R as in the right of Figure 5b, and $\chi_1 = (u_0, w, v_0)$. Also, if l (resp. l') is the $|{+}\rangle$-link (resp. $|{-}\rangle$) in R, let n (resp. m) be its level, and let $a = \mathsf{CID}(l)$ for some $a \in \mathfrak{C}^*$ (resp. b). By definition of context assignment,

$$\begin{aligned}\mathfrak{c}(\pi) &= \mathfrak{c}(\pi_0 :: (u_0, w, v_0) :: \pi_1) &(123)\\ &= \mathfrak{c}(\pi_0) \cdot !^n a \cdot !^m\overline{b} \cdot \mathfrak{c}(\pi_1) &(124)\end{aligned}$$

By definition of reduction, and in particular by (17):

$$\begin{aligned}\rho(\pi) &= \rho(\pi_0 :: (u_0, w, v_0) :: \pi_1) &(125)\\ &= \{\pi_0 :: (u_0, z_{0_0}, v_0) :: \pi_0\} &(126)\end{aligned}$$

So let π' be the unique path in $\rho(\pi)$. If we consider its context assignment, we easily conclude by definition of lifting:

$$\begin{aligned}\mathfrak{c}(\pi') &= \mathfrak{c}(\pi_0 :: (u_0, z_{0_0}, v_0) :: \pi_1) &(127)\\ &= \mathfrak{c}(\pi_0) \cdot !^m\overline{b} \cdot !^n a \cdot \mathfrak{c}(\pi_1) &(128)\\ &= \mathfrak{c}(\pi). &(129)\end{aligned}$$

11. Rule (a) and $n = 1$. Let R as in left of Figure 5b. If h, h' are the lifts within R, then let: $m = \ell(h) = \ell(h')$, $a = \mathsf{CID}(h)$, and $a' = \mathsf{CID}(h')$. By definition of context assignment,

$$\begin{aligned} \mathfrak{c}\,(\pi) &= \mathfrak{c}\,(\pi_0 :: (u_0, w, v_0) :: \pi_1) && (130)\\ &= \mathfrak{c}\,(\pi_0) \cdot !^m a \cdot !^m \overline{a'} \cdot \mathfrak{c}\,(\pi_1)\,. && (131)\end{aligned}$$

But as per Proposition 56 we know that $\mathfrak{c}\,(\pi) >_{\ell(v)}$. Therefore, it must be the case that $\mathfrak{c}\,(\pi_0) >_{\ell(v)}$ and $a \cdot \overline{a'} = 1$, which means that $a = a'$. Hence,

$$= \mathfrak{c}\,(\pi_0) :: \mathfrak{c}\,(\pi_1)\,. \qquad (132)$$

Now, by definition of reduction (cf. (16)), we have that

$$\begin{aligned} \rho(\pi) &= \rho(\pi_0 :: (u_0, w, v_0) :: \pi_1) && (133)\\ &= \{\pi_0 :: (u_0 = v_0) :: \pi_1\}. && (134)\end{aligned}$$

Let $\pi' \in \rho(\pi)$ and observe that context of π' is equivalent to that of π.

$$\mathfrak{c}\,(\pi') = \mathfrak{c}\,(\pi_0) :: \mathfrak{c}\,(\pi_1)\,. \qquad (135)$$

12. Rule (m) and $n = 1$. Assume R as in the right of Figure 5h, and let $\chi_1 = (v_0, u_i, w)$, for some $0 \le i \le h$. Also, let n be the level of the ?-link and of the ⊦)-link in R, say l. Finally, let $a = \mathsf{CID}(l)$ for some $a \in \mathfrak{C}^*$. By definition of context assignment, and by definition of the restriction operator:

$$\begin{aligned} \mathfrak{c}\,(\pi) &= \mathfrak{c}\,(\pi_0 :: (v_0, u_i, w) :: \pi_1) && (136)\\ &= (\mathfrak{c}\,(\pi_0) \cdot !^n a)\,|_n \cdot \mathfrak{c}\,(\pi_1) && (137)\\ &= (\mathfrak{c}\,(\pi_0))\,|_n \cdot \mathfrak{c}\,(\pi_1)\,. && (138)\end{aligned}$$

By definition of reduction, and in particular by (23):

$$\begin{aligned} \rho(\pi) &= \rho(\pi_0 :: (v_0, u_i, w) :: \pi_1) && (139)\\ &= \{\pi_0 :: (v_0, w) :: \pi_0\}. && (140)\end{aligned}$$

Consider π' is the unique path in $\rho(\pi)$, and look at its context assignment. We conclude.

$$\begin{aligned} \mathfrak{c}\,(\pi') &= \mathfrak{c}\,(\pi_0 :: (v_0, w) :: \pi_1) && (141)\\ &= (\mathfrak{c}\,(\pi_0) \cdot !^n a)\,|_m \cdot \mathfrak{c}\,(\pi_1) && (142)\\ &= \mathfrak{c}\,(\pi) && (143)\end{aligned}$$

◂

Proof of Proposition 56 (Strong positivity). Let $U \in \mathsf{UG}$, and $\pi : a \rightsquigarrow b$ be a path such that a is a !-premiss or the root of U. If $U \in \mathsf{EPN}$, then the claim holds trivially, since $\mathfrak{c}\,(\pi) = 1 > 0$. So suppose otherwise that $U = \rho(U')$, for some $U' \in \mathsf{UG}$ and some UG- or UR-reduction step ρ. Also, let R be the redex of ρ, and let $\rho(R)$ be the reduct of R.

1. If a, b do not belong to $\mathsf{int}(R)$, then there exists $\pi' : a \rightsquigarrow b$ path of U that is long enough for R and such that $\pi = \rho(\pi')$. Now a is a !-premiss or the root of U', so per IH $\mathfrak{c}\,(\pi') >_{\ell(a)} 0$. But by invariance Lemma 57 we have $\mathfrak{c}\,(\pi') = \mathfrak{c}\,(\pi)$, hence $\mathfrak{c}\,(\pi) >_{\ell(a)} 0$.

2. If a or b belong to $\mathsf{int}(R)$, we distinguish some cases depending on the rule of R. When needed to avoid confusion between paths, we shall precise as a subscript the graph to which they belong, e.g. π_U or $\pi'_{U'}$.

   a. Rules $(!), (\multimap), (a), (m)$. Absurd: by inspection of the definition we verify that $\mathsf{int}(R) = \varnothing$.

   b. Rule $(d!)$. Let R be as in Figure 5e (left), and observe that $\mathsf{int}(R) = \{z_0\}$.

      i. If $a = z_0$, then $\pi = (z_0, u)_U :: \eta$ for some downward path η. We notice immediately that $\mathfrak{c}\left((z_0, u)_{U'}\right) > 0$, by definition of contexts, as well as $\mathfrak{c}\left(\eta\right) >_{\ell(a)} 0$, by inductive hypothesis. Hence $\mathfrak{c}\left(\pi\right) >_{\ell(a)}$.

      ii. If $b = z_0$, then $\pi = \eta :: (v_0, z_0)_U$ for some downward path η. But $\mathfrak{c}\left(\eta\right) >_{\ell(z_0)} 0$, while $\mathfrak{c}\left((v_0, z_0)\right) = 1 > 0$. Therefore, $\mathfrak{c}\left(\pi\right) >_{\ell(a)} 0$.

   c. Rule $(d?)$. Let R be as in Figure 5f (right), and observe that $\mathsf{int}(R) = \{z_{0_0}, \dots, z_{0_k}\}$, so let us consider z_{0_i} for some $0 \le i \le k$.

      i. If $a = z_{0_i}$, then we absurdly contradict the hypothesis of a being a !-premiss or the root of U.

      ii. If $b = z_{0_i}$, then $\pi = \eta :: (u_i, z_{0_i})_U$ for some downward path η. Inspecting ρ we observe that $\mathfrak{c}\left((u_i, z_{0_i})_U\right) = \mathfrak{c}\left((u_i, w, v_0)_{U'}\right)$. Also, by IH we have that $\mathfrak{c}\left(\eta :: (u_i, w, v_0)\right) >_{\ell(a)} 0$, hence $\mathfrak{c}\left(\pi\right) >_{\ell(a)} 0$.

   d. Rules $(d\overset{+}{\multimap})$. Let R be as in Figure 5c (left), and notice that $\mathsf{int}(R) = \{z_{1_0}, z_{2_0}\}$. Also, since $\rho(R)$ does not include a !-link, $a \notin \mathsf{int}(R)$.
   Now, if $b = z_{2_0}$, then the situation is exactly as in the previously discussed case 2.b.ii. So assume $b = z_{1_0}$, and let $\chi_1 = (v_0, x_{2_0}, u_2)$, and $\chi_2 = (u_1, z_{1_0})$. Clearly, χ_2 is a suffix of π. What about χ_1?

      i. If $\chi_1 \not\subset \pi$, then observe that $\ell(a) > \ell(z_{1_0})$. Let $\pi = \pi_0 :: \chi_2$, Now, by definition $\mathfrak{c}\left((u_1, z_{1_0})\right) =_{\ell(a)} 1$, and $\pi_{0U'} >_{\ell(a)} 0$ per inductive hypothesis. Hence the claim.

      ii. If $\chi_1 \subset \pi$, then let $\pi = \pi_0 :: \chi_1 :: \pi_1 :: \chi_2$, and consider $\pi' = \pi_{0U'} :: \chi'_1 :: \pi_{1U'}$, where $\chi'_1 = (v_0, w, u_2)$. Also, let n be the level of the $\langle\!\ast\rangle$-links crossed in χ_1, χ_2, and consider the context of π at level $m \in \mathbb{N}$.

         A. If $m \neq n$, as in last sub-case we immediately obtain that $\mathfrak{c}\left(\pi\right)(m) = \mathfrak{c}\left(\pi'\right)(m)$. But by IH $\mathfrak{c}\left(\pi'\right)(m) > 0$, thus $\mathfrak{c}\left(\pi\right)(m) > 0$.

         B. Otherwise $m = n$. In U' we know by IH that $\mathfrak{c}\left(\pi_0 :: \chi'_1\right) =_{\ell(a)} \mathfrak{c}\left(\pi_0 :: \chi'_1 :: \pi_1\right)$, hence $\mathfrak{c}\left(\pi_0 :: \chi'_1\right)(n) = \mathfrak{c}\left(\pi_0 :: \chi'_1 :: \pi_1\right)(n)$. Because of this, we can apply Lemma 54, to obtain that, for any $m' \in \mathbb{N}$ (hence in particular for $m' = m$), there exist $c, d \in \mathfrak{C}^*$ such that $\mathfrak{c}\left(\pi_0 :: \chi'_1\right)(m') = c \cdot d$ and $\mathfrak{c}\left(\pi_1\right)(m') = \overline{d} \cdot d$, where d is positive.
         Let $e = \mathfrak{c}\left(\chi'_1\right)(m)$, and observe that $\mathfrak{c}\left(\chi_1\right) = \mathfrak{c}\left(\chi'_1\right)$. Also, notice that e has to be the rightmost object of the stable form of $\mathfrak{c}\left(\pi_0 :: \chi'_1\right)(m')$.

            I. If $d = 1$, then $\mathfrak{c}\left(\pi_0 :: \chi'_1\right)(m') = c' \cdot e$, and $\mathfrak{c}\left(\pi_1\right)(m') = 1$. So we can write the n-context of π as follows.

$$
\begin{aligned}
\mathfrak{c}\left(\pi\right)(n) &= \mathfrak{c}\left(\pi_0 :: \chi_1 :: \pi_1 :: \chi_2\right) && (144)\\
&= \mathfrak{c}\left(\pi_0 :: \chi_1\right)(n) \cdot \mathfrak{c}\left(\pi_1\right)(n) \cdot \mathfrak{c}\left(\chi_2\right)(n) && (145)\\
&= c' \cdot e \ \cdot \ 1 \ \cdot \ \overline{e} && (146)\\
&= c' && (147)\\
&= \mathfrak{c}\left(\pi_0\right). && (148)
\end{aligned}
$$

II. If $d \neq 1$, then $\mathfrak{c}(\pi_0 :: \chi'_1)(m') = c \cdot d' \cdot e$. and $\mathfrak{c}(\pi_1)(m') = \overline{e} \cdot \overline{d'} \cdot d' \cdot e$. Therefore, the context at level n of π is:

$$\mathfrak{c}(\pi)(n) = \mathfrak{c}(\pi_0 :: \chi_1 :: \pi_1 :: \chi_2) \tag{149}$$
$$= \mathfrak{c}(\pi_0 :: \chi_1)(n) \cdot \mathfrak{c}(\pi_1)(n) \cdot \mathfrak{c}(\chi_2)(n) \tag{150}$$
$$= c \cdot d' \cdot e \quad \cdot \quad \overline{d' \cdot e} \cdot d' \cdot e \quad \cdot \quad \overline{e} \tag{151}$$
$$= c \cdot d' \cdot e \quad \cdot \quad \overline{e} \cdot \overline{d'} \cdot d' \cdot e \quad \cdot \quad \overline{e} \tag{152}$$
$$= c' \cdot d' \cdot e \tag{153}$$
$$= \mathfrak{c}(\pi_0). \tag{154}$$

But by IH $\mathfrak{c}(\pi_0)(n) > 0$, thus $\mathfrak{c}(\pi)(n) > 0$.

Therefore the claim.

e. Rules $(d\bar{\multimap}), (r?), (r\bar{\multimap}), (s)$. Since $\rho(R)$ does not contain an !-link, $a \notin \mathsf{int}(R)$, so it must be the case that $b \in \mathsf{int}(R)$. Then we can follow both arguments used in cases 2.b.ii and 2.c.ii.

f. Rule (tD). Let R be as in Figure 6, and observe that in this case we have $\mathsf{int}(R) = \bigcup_{0 \leq i \leq h} V(b_i)$, where we recall that $V(b)$ is the set of vertices of the i-th copy of the box b.

i. If $a, b \in \bigcup_{0 \leq i \leq h} V(b_i)$, then observe that any vertex of π has a unique anti-residual in U'. So let $\pi' : a' \rightsquigarrow b'$ be the path built by these anti-residuals following the same order of π. Unsurprisingly, $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$. But $\mathfrak{c}(\pi') >_{\ell(a')} 0$ by IH, and $\ell(a') = \ell(a)$, therefore $\mathfrak{c}(\pi) >_{\ell(a)} 0$.

ii. If $a \in \bigcup_{0 \leq i \leq h} V(b_i) \not\ni b$, then let $\pi = \gamma :: (U_{j_i}, u_j) :: \eta$, for some $0 \leq i \leq h$ and $1 \leq j \leq k$. Now, consider $\pi' = \gamma :: (U_{j_i}, u_j) :: \eta$. Similarly to the previous case 2.f.i, we have $\mathfrak{c}(\pi') = \mathfrak{c}(\pi)$, and $\mathfrak{c}(\pi') >_{\ell(a)} 0$, hence the claim.

iii. If $a \notin \bigcup_{0 \leq i \leq h} V(b_i) \ni b$, then let $\pi_U = \pi_0 :: (v_j, w_j) :: \gamma_j$ for some $1 \leq j \leq k$, and consider $\pi'_{U'} = \pi_0 :: (v_j, z, w) :: \gamma$. Now, firstly observe that $\mathfrak{c}(\pi_{0U}) = \mathfrak{c}(\pi_{0U'})$, which is positive by IH. Secondly, we have $\mathfrak{c}((v_j, z, w)) > 0$ by definition. Finally, we notice that $\mathfrak{c}(\gamma_j) = \mathfrak{c}(\gamma)$, which is positive by IH. (More details are comprehensively explained in case 3 of the proof of Lemma 57.) Therefore π has a positive weight.

◂

Proof of Lemma 58 (Lambda-compatibility). Let $U \in \mathsf{UG}$, $\pi : a \rightsquigarrow c$ and $\phi : c \rightsquigarrow d$ be two paths such that a is a !-premiss or the root of U, and there exists $(d, c \overset{+}{\multimap} b)$. If U is a levelled proof-net, then the claim immediately holds, since $\mathfrak{c}(\pi) = \mathfrak{c}(\phi) = 1 = \mathfrak{c}(\pi :: \phi)$. Suppose otherwise that $U = \rho(U')$, for some $U' \in \mathsf{UG}$ and some UG- or UR-reduction step ρ. Also, let R be the redex of ρ, and let $\rho(R)$ be the reduct of R.

1. If a, c, and d do not belong to $\mathsf{int}(R)$, then there exist $\pi' : a \rightsquigarrow c$ and $\phi' : c \rightsquigarrow d$ paths of U that are long enough for R. Now, a is a !-premiss the root of U', so per IH $\mathfrak{c}(\pi') =_{\ell(a)} \mathfrak{c}(\pi' :: \phi')$. But by invariance Lemma 57, $\mathfrak{c}(\pi') = \mathfrak{c}(\pi)$ and $\mathfrak{c}(\phi') = \mathfrak{c}(\phi)$. Hence $\mathfrak{c}(\pi) >_{\ell(a)} 0$.
2. If a, c, or d belongs to $\mathsf{int}(R)$, we distinguish some cases depending on the rule of R. Subscripts of paths denotes the graph to which they belong, e.g. $\pi_U, \pi'_{U'}$.

a. Rules $(!), (\multimap), (a), (m)$. Absurd: by inspection of the definition we verify that $\mathsf{int}(R) = \varnothing$.

b. Rules $(d?), (d\multimap^{-}), (r\multimap^{-}), (r?), (s)$. Then $\rho(R)$ does not contain any link of kind ! or $\multimap^{+}$, so a, c, and d cannot be in $\mathsf{int}(R)$. Absurd.
c. Rule $(d!)$. Let R be as in Figure 5e (left), and observe that $\mathsf{int}(R) = \{z_0\}$. Since we assumed c, d being respectively the second and the first premiss of a $\multimap^{+}$-link, it must be the case that $c, d \neq z_0$, which implies that $a = z_0$. So, let $\pi = (z_0, u) :: \eta$ for some downward path η. By IH, we know that $\mathfrak{c}(\eta) =_{\ell(a)} \mathfrak{c}(\eta :: \phi)$. Hence $\mathfrak{c}(\pi) >_{\ell(a)} \mathfrak{c}(\pi :: \phi)$.
d. Rule $(d\multimap^{+})$. Let R be as in Figure 5c (left). We preliminarily notice that $\mathsf{int}(R) = \{z_{1_0}, z_{2_0}\}$. Also, since $\rho(R)$ does not include a !-link, $a \notin \mathsf{int}(R)$. Moreover, it must the case that both $c, d \in \mathsf{int}(R)$. because $\rho(R)$ contains both the premisses of a $\multimap^{+}$-link. In particular, this mean that $c = z_{2_0}, d = z_{1_0}$. Then the claim has been proven in the proof of Proposition 56, case 2.d.ii (in particular Equation (148) and 154).
e. Rule (tD). Let R be as in Figure 6, where $\mathsf{int}(R) = \bigcup_{0\le i\le h} V(B_i)$. By hypothesis on c, d we have that c belongs to $\mathsf{int}(R)$ if and only if d does so.
   i. If $a, c, d \in \bigcup_{0\le i\le h} V(b_i)$, then observe that any vertex of π has a unique anti-residual in U'. So let $\pi' : a' \rightsquigarrow c'$ and $\phi' : c' \rightsquigarrow d'$ be the path built by these anti-residuals following the same order of π and ϕ, respectively. Then $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$ and $\mathfrak{c}(\phi) = \mathfrak{c}(\phi')$. By IH we have $\mathfrak{c}(\pi') =_a \mathfrak{c}(\pi' :: \phi')$, therefore $\mathfrak{c}(\pi) =_a \mathfrak{c}(\pi :: \phi)$.
   ii. If $a \in \bigcup_{0\le i\le h} V(b_i) \not\ni c, d$, then let $\pi = \gamma :: (U_{j_i}, u_j) :: \eta$ for some $0 \le i \le h$ and $1 \le j \le k$. Now, consider $\pi' = \gamma :: (U_{j_i}, u_j) :: \eta$, and observe that $\mathfrak{c}(\pi') = \mathfrak{c}(\pi)$. Now, by IH we know $\mathfrak{c}(\pi') =_a \mathfrak{c}(\pi' :: \phi)$, so we conclude that $\mathfrak{c}(\pi) =_a \mathfrak{c}(\pi :: \phi)$, quod erat demonstrandum.
   iii. If $a \notin \bigcup_{0\le i\le h} V(b_i) \ni c, d$, then let $\pi = \pi_0 :: (v_j, w_j)_U :: \gamma_j$, for some $1 \le j \le k$. Now, consider $\pi' = \pi_0 :: (v_j, z, w)_{U'} :: \gamma$. Let m be the level of the !-link in R. Now, firstly observe that $\mathfrak{c}(\pi_{0U}) = \mathfrak{c}(\pi_{0U'})$ and $\mathfrak{c}(\gamma_j) = \mathfrak{c}(\gamma)$. Moreover, by definition of context assignment, $\mathfrak{c}(\pi_0)(m) = 1$. Therefore $\mathfrak{c}(\pi')(m) = \mathfrak{c}(\gamma)(m)$. But $\mathfrak{c}(\pi')(m) = \mathfrak{c}(\pi' :: \phi)(m)$, thus $\mathfrak{c}(\gamma)(m) = \mathfrak{c}(\gamma :: \phi)(m)$. Secondly, we have $\mathfrak{c}((v_j, w_j)) > 0$ per definition. Thirdly, applying positivity Proposition 56 as IH, we obtain that $\mathfrak{c}(\gamma) >_{\ell(a)} 0$, hence $\mathfrak{c}(\gamma_j) >_{\ell(a)} 0$. Hence the claim. (See also case 3 of the proof of Lemma 57.)

◀

Proof of Lemma 59 (Box-compatibility). The proof is very similar to that of Lemma 59, so we provide only an outline to avoid pedantry. One goes by induction on the length of the reduction sequence from a levelled proof-net, where the statement holds trivially, to the given UG. Given a step ρ, Lemma 57 allows to obtain the claim by IH in the case of paths whose counter-image with respect to ρ is long enough for the redex of ρ. Otherwise, one proceeds with an inspection of possible redexes whose residuals are crossed by the paths $\pi : u \rightsquigarrow v$ and $pi' : u \rightsquigarrow v'$ of interest.

- One interesting case is that of the (tD). Its behaviour is irrelevant at levels strictly smaller than $\ell(u)$ and greater or than $\ell(v)$, which are out of the statements' scope. In between such interval, the rule adds a lift in front of one of the copies of a box B, which carry a $\mathfrak{C}^*$ object to the paths crossing it. It is sufficient to observe that for any box B' at level strictly smaller than $\ell(v)$, $v \in B'$ if and only if $v' \in B'$ (such property comes from a crucial feature of IEAL: a ?-premiss belongs at most to one box); and to recall the box connectivity Item 4. This allow to observe that any new lift generated by a (tD)- step on B necessarily introduces a $\mathfrak{C}^*$ object which belongs to both the contexts of π, π'
- The only other notable case is $(r?)$. Assume it happens on v and involve a lift having variable occurrence c at level n, which therefore is the only one to be considered. By IH,

both the n-th contexts of the anti-residuals of v, v' have suffix c. Therefore while c is not the suffix of the n-th context of v, the other premiss v' has $c \cdot c'$, so they are equivalent. ◂

Proof of Lemma 60 (Unary Contraction). Let $\pi : u \rightsquigarrow v$ be a downward path in a $U \in \mathsf{UG}$, where u is a !-premiss, v is a ?-premiss, and both belong to a box B. If U is a levelled proof-net, then the claim holds trivially, since $\mathfrak{c}(\pi) = 1 > 0$. So suppose otherwise that $U = \rho(U')$, for some $U' \in \mathsf{UG}$ and some UG- or UR-reduction step ρ. Let R be the redex of ρ, and $\rho(R)$ its residual.

1. If u and v do not belong to $\mathsf{int}(R)$, then there exists $\pi' : u \rightsquigarrow v$ path of U that is long enough for R and such that $\pi = \rho(\pi')$. Now, a is a !-premiss or the root of U', so per IH $\mathfrak{c}(\pi) = 1$. But by invariance Lemma 57 we have $\mathfrak{c}(\pi') = \mathfrak{c}(\pi)$, and in particular $\mathfrak{c}(\pi')(\ell(u)) = \mathfrak{c}(\pi)(\ell(u))$ hence $\mathfrak{c}(\pi')(\ell(u)) = 1$.
2. Otherwise u or v belongs to $\mathsf{int}(R)$.
   a. Rules $(!), (\multimap), (d\overset{+}{\multimap}), (d\overset{-}{\multimap}), (a), (s), (r\overset{-}{\multimap})$. Absurd: by inspection of the definition we verify that $\mathsf{int}(R) = \varnothing$.
   b. Rules $(d!)$. Let $\pi = \chi :: \eta$, and $\pi' = \chi' :: \eta$ such that π' is long enough for R and $\rho(\pi') = \pi$.
   Now, by definition of UG, the level of the involved lift has to be strictly smaller than n. Therefore $\mathfrak{c}(\chi')(\ell(u)) = \mathfrak{c}(\chi)(\ell(u))$. But by IH $\mathfrak{c}(\eta)(\ell(u))$, hence we conclude.
   c. Rules $(d?), (r?)$. These cases are duals of 2b, and omitted.
   d. Rule (m). Let $\pi = \eta :: \chi$, and $\pi' = \eta :: \chi'$ such that π' is long enough for R and $\rho(\pi') = \pi$. Consider the $\langle\!+\!\rangle$-link l in R, and observe that by definition, $\mathfrak{c}(\chi')(\ell(u)) = \mathsf{CID}(l) \neq 1$. Hence $\mathfrak{c}(\pi')(\ell(u)) =\neq 1$. But this is the negation of IH. Absurd.
   e. Rule (tD). Let R be as in Figure 6, and say the box of R is named C, and the copies of C are named $C_0 \dots C_h$, for some $h > 0$. In this case $\mathsf{int}(R) = \bigcup_{0 \le i \le h} V(C_i)$. We distinguish three cases depending on the level n of the box C.
      i. If $n > \ell(u)$, then in U we have that $C_i \subset B$ for any $0 \le i \le h$. This implies that there exists a downward path π' in U' such that $\rho(\pi') = \pi$. But this contradicts our assumption that u or v belong to $\mathsf{int}(R)$. Absurd.
      ii. If $n = \ell(u)$, then there exists $0 \le i \le h$ such that $C_h = B$. Look again at Figure 6 and observe that $v = U_{j_i}$ for some $1 \le j \le k$, that is the residual of the ?-premiss U_j in U'. But U_j has a total of $h+1$ residual vertices in U, all being premiss of the same ?-link l. But this contradicts our hypothesis that v is the only premiss of l. Thus there is nothing to be proven here.
      iii. If $n < \ell(u)$, then there exists $0 \le i \le h$ such that $C_i \supset B$. Notice that π is one of the $h+1$ copies of some path π' of U', i.e. $\pi \in \rho(\pi)$, for some downward path π of U'. Thus, $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$, from which the claim.

◂

Proof of Proposition 29. Immediate from the Proposition 56. ◂

A.3.3 Path irrelevance for l-contexts

We now show that the contexts of rooted paths to a given vertex does not depend on the choice of path, which justifies the generalisation of the notion of l-contexts to be about vertices.

▸ **Lemma 61** (Strong path irrelevance). *Let $\pi : u \rightsquigarrow v$ and $\pi' : u \rightsquigarrow v$ be two downward paths in a $U \in$ UG where u is the root of U or a premiss of a !-link. Then $\mathfrak{c}(\pi) =_{\ell(u)} \mathfrak{c}(\pi')$.*

Proof. Let $U \in$ UG, and let $\phi, \phi' : w \rightsquigarrow v$ be two paths in U. We go by induction on the number e of vertices belonging to both paths.

1. If $e = 1$, then $\phi = \phi'$ and the claim is trivially verified.
2. Otherwise $e > 1$, so let $\phi = \gamma :: \pi$ and $\phi' = \gamma' :: \pi'$ such that: $\gamma, \gamma' : w \rightsquigarrow u$ and $\pi, \pi' : u \rightsquigarrow v$, where $u \neq v$. Since by definition $\mathfrak{c}(\phi) = \mathfrak{c}(\gamma) \cdot \mathfrak{c}(\pi)$ and $\mathfrak{c}(\phi') = \mathfrak{c}(\gamma') \cdot \mathfrak{c}(\pi')$, where per IH we have $\mathfrak{c}(\gamma) = \mathfrak{c}(\gamma')$, we have just reduced the claim to prove $\mathfrak{c}(\pi) = \mathfrak{c}(\pi')$.

   By construction, v has to be the o vertex of a link having at least two ι vertices. There are only two links of such kind: $\overset{+}{\multimap}$ and ?. But the first premiss of a $\overset{+}{\multimap}$-link cannot be reached from its second premiss with an upward path. Therefore v is the conclusion of a ?-link l.
   Let z, z' respectively be the two premisses of l such that $\pi = \delta :: (z, v)$ and $\pi' = \delta' :: (z', v)$. By Lemma 59, $\mathfrak{c}(\delta) =^{\ell(w)}_{\ell(u)} \mathfrak{c}(\delta')$. And by definition of context assignment, $\mathfrak{c}(\pi) = (\mathfrak{c}(\delta))|_{\ell(w)}$ and $\mathfrak{c}(\pi') = (\mathfrak{c}(\delta'))|_{\ell(w)}$. Hence the claim.

◂

Proof of Proposition 30. Immediate from Lemma 61. ◂

▸ **Lemma 62.** *Given a $U \in$ UG and a reduction step ρ on a redex R, if $v \notin \mathsf{int}(R)$, then $\mathfrak{c}(v) = \mathfrak{c}(v')$, for any residual v' of v.*

Proof. By hypothesis any downward path π from the root r of U to v is long enough for R. By definition, there exists $\pi' \in \rho(\pi)$ such that $\pi' : r \rightsquigarrow v'$. Moreover, by long invariance Lemma 57, we have $\mathfrak{c}(\pi) = \mathfrak{c}(\pi') = \mathfrak{c}(v')$. ◂

A.4 Complexity analysis

A.4.1 Unshared metrics

First of all we provide the formal account of the various metrics stated in Lemma 35, which can be done with a bit of patience.

Proof of Lemma 35 (Metrics on UG-reduction). Given $U \in$ UG, let R be a redex in U, and let μ be the reduction step on R. We proceed with a case analysis depending on the kind of redex.

1. Rule ($\multimap$). Let R as in Figure 4a.
   a. If $w \notin \mathsf{Sh}(U)$, then $v_1, v_2, w, u_1, u_2 \notin \mathsf{Sh}(U)$ and $v_1, v_2 \notin \mathsf{Sh}(\mu(U))$. No boundary share components are changed by μ. Hence, trivially:

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{155}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 9 - 0 = 9 \tag{156}$$

$$\Delta\mathsf{BShC}(\mu) = 0 \tag{157}$$

   b. If $w \in \mathsf{Sh}(U)$, we separately consider three portions of the redex.
      - Consider w and the two links, which all belong to $\mathsf{iSh}(U)$, so they contributes with $\Delta\mathsf{iSh}(\mu) = -1 - 2 \times 3 = -7$.

- Consider u_1, v_1, whose residual is $u_1 = v_1$. Observe that $v_1 \notin \mathsf{bSh}(U)$. We now distinguish six sub-cases.
  i. If $u_1, v_1 \in \mathsf{iSh}(U)$, then $u_1 \in \mathsf{iSh}(\mu(U))$. Therefore $\Delta\mathsf{iSh}(\mu) = -1$ and $\Delta\mathsf{BShC}(\mu) = 0$.
  ii. If $u_1 \in \mathsf{bSh}(U)$ and $v_1 \in \mathsf{iSh}(U)$, then $u_1 \in \mathsf{bSh}(\mu(U))$. Therefore $\Delta\mathsf{iSh}(\mu) = -1$ and $\Delta\mathsf{BShC}(\mu) = 0$.
  iii. If $u_1 \in \mathsf{iSh}(U)$ and $v_1 \in \mathsf{BSh}(U)$, then $u_1 \in \mathsf{BSh}(\mu(U))$. Hence, $\Delta\mathsf{iSh}(\mu) = -1$ and $\Delta\mathsf{BShC}(\mu) = 0$.
  iv. If $u_1 \in \mathsf{bSh}(U)$ and $v_1 \in \mathsf{BSh}(U)$, then by definition of pseudo-boundary, there exists a path $\pi_i : x_i \rightsquigarrow u_i$ such that $x_i \in \mathsf{iSh}(U)$ while $x'_i \in \mathsf{bSh}(U)$ for every $x'_i \neq x_i$ in π_i. In the residual we have not only $u_1 \in \mathsf{BSh}(\mu(U))$, but also $x_i, x'_i \in \mathsf{BSh}(\mu(U))$. Hence $\Delta\mathsf{iSh}(\mu) = -1$ and $\Delta\mathsf{BShC}(\mu) = 0$.
  v. If $u_1 \in \mathsf{BSh}(U)$ and $v_1 \in \mathsf{BSh}(U)$, then $u_1 \in \mathsf{BSh}(\mu(U))$. Therefore $\Delta\mathsf{iSh}(U) = 0$. Moreover $v_1 = u_1$ form a new boundary share component, so $\Delta\mathsf{BShC}(\mu) = 1$.
  vi. If $u_1 \in \mathsf{BSh}(U)$ and $v_1 \in \mathsf{iSh}(U)$, then $u_1 \in \mathsf{BSh}(U)$. Thus, $\Delta\mathsf{iSh}(U) = -1$. and $\Delta\mathsf{BShC}(\mu) = 0$.
- Consider u_2, v_2, whose residual is $u_2 = v_2$. The analysis is identical to the previous case, where u_2 plays the role of v_1 and v_2 that of u_1.

Summing up, we obtain what follows.

$$\Delta\mathsf{iSh}(\mu) = -9 + \Delta\mathsf{BShC}(\mu) \tag{158}$$

$$\begin{aligned}\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) &= 9 - \big(-9 + \Delta\mathsf{BShC}(\mu)\big)\\ &= 18 - \Delta\mathsf{BShC}(\mu)\end{aligned} \tag{159}$$

$$\Delta\mathsf{BShC}(\mu) \in [0, 2] \tag{160}$$

2. Rule (!). Let R as in Figure 5a.

a. Assume that $w \notin \mathsf{Sh}(U)$. We immediately notice that $u \notin \mathsf{Sh}(U)$, but we can easily observe also that $v_0 \notin \mathsf{Sh}(U)$. Suppose otherwise that v_0 is shared and let $m \in \mathbb{N}$ such that $\mathfrak{c}(v_0)(m) \not\preceq 1$.
  i. If $m \neq \ell(u)$ then, by definition of context assignment, we would have also $\mathfrak{c}(w)(m) \not\preceq 1$, contradicting our hypothesis.
  ii. Otherwise, $m = \ell(u)$, and let $\mathfrak{c}(v_0) = x_{i:m'}$ with $m' \geq i > 0$. Now, this would absurdly imply that the ?-link of R has at least $i + 1$ premisses, while it has only 1 (i.e. v_0).

Therefore $v_0 \notin \mathsf{Sh}(U)$, which implies:

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{161}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 6 - 0 = 6 \tag{162}$$

$$\Delta\mathsf{BShC}(\mu) = 0 \tag{163}$$

b. Otherwise, $w \in \mathsf{Sh}(U)$ and the analysis follows almost identical to previous sub-case 1b.

$$\Delta\mathsf{iSh}(\mu) = -6 \tag{164}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 6 - (-6) = 12 \tag{165}$$

$$\Delta\mathsf{BShC}(\mu) \in [0, 1] \tag{166}$$

3. Rule (tD). Let R be as in Figure 6.

a. Suppose first that $z \notin \mathsf{Sh}(U)$.

The vertices $v_0, \ldots, v_h, V_1 \ldots, V_k, u_1, \ldots, u_k$ are irrelevant with respect to the interior share, since none of them cannot belong to $\mathsf{Sh}(\mu(U))$. The same goes for z, since $w \notin \mu(U)$.

B_0 is irrelevant as well. Indeed, the newly introduced master lift cannot affect $\Delta\mathsf{iSh}(\mu)$ nor $\Delta\mathsf{BShC}(\mu)$. Moreover, observe that in general, if p is the first premiss of a $?^n$-link, then $\mathfrak{c}\,(p)\,(l)$ is master. Hence, by our hypothesis on z, we also have $v_0 \notin \mathsf{Sh}(U)$.

Now let $1 \le i \le h$, and let y_i be a vertex in $B_i \subset \mu(U)$ residual of y in $B \subset U$. We observe that $y_i \in \mathsf{iSh}(\mu(U))$ if $y \neq w$ and $y \in \mathcal{E}(U)$. Therefore:

$$\Delta\mathsf{iSh}(\mu) = h \times (\#\mathcal{E}(B) - 1) \tag{167}$$

$$\begin{aligned} \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) &= h \times \#\mathcal{E}(B) + 2h + 4 - \big(h \times \#\mathcal{E}(B) - h\big) \\ &= 3h + 4 \end{aligned} \tag{168}$$

Now let us consider share boundaries. Clearly we have h new lifts of this kind. What about the variation of the number of share components? Notice for any $c \in \mathsf{BShC}(B)$ and any c_i copy of c, we have c_i is not a boundary share component, but a set of pseudo-boundary share vertices. So the only way for μ to create a new boundary share component involve the new lifts, which are boundary because we assumed $w \notin \mathsf{Sh}(U)$. Since by Proposition 56 in B there cannot be $(s \mid\!\!-\!\!\rangle^n\ t)$ such that $w \rightsquigarrow t$, the only other way to have a boundary share component is that $V(B) = \{w\}$. So if this is case we have $\Delta\mathsf{BShC}(\mu) = h$, otherwise $\Delta\mathsf{BShC}(\mu) = 0$:

$$\Delta\mathsf{BShC}(\mu) = \{0, h\} \tag{169}$$

b. Otherwise $z \in \mathsf{Sh}(U)$. We separately discuss subsets of vertices of the redex.

- Consider z and the two main links. By this last hypothesis $z \in \mathsf{iSh}(U)$, while $z \notin \mu(U)$, so in this portion of the redex we have $\Delta\mathsf{iSh}(\mu) = -1 - (h+2) - 2 = -h - 5$.
- Consider $V(B) \setminus w$, and let $x \in V(B) \setminus w$ and $x_i \in V(B_i) \setminus w_i$ for some $0 \le i \le h$. If $i = 0$ then nothing changes with respect to the share positioning: x belongs to $\mathsf{Sh}(B)$, $\mathsf{iSh}(B)$, $\mathsf{bSh}(B)$, or $\mathsf{BSh}(B)$ if and only if x_0 respectively belongs to $\mathsf{Sh}(B_0)$, $\mathsf{iSh}(B_0)$, $\mathsf{bSh}(B_0)$, or $\mathsf{BSh}(B_0)$. The same goes for boundary lifts: $l \in \mathsf{bLft}(B)$ if and only if $l' \in B_0$, where l' is the residual of l. If instead $i > 0$, we first observe that if $x \in \mathsf{BSh}(U)$, then $x_i \in \mathsf{bSh}(\mu(U))$, and that any lifts in $L(B_i)$ are interior. Hence, in both cases, in this portion of the redex we have that $\Delta\mathsf{iSh}(\mu) = h \times (\#\mathcal{E}(B) - 1)$, while $\Delta\mathsf{BShC}(\mu) = 0$.
- Consider w and v_i with $0 \le i \le h$. Since $z \in \mathsf{Sh}(U)$, it must be the case that $w, v_i \in \mathsf{Sh}(U)$. Also, notice that, by definition of pseudo-boundaries, $w \notin \mathsf{bSh}(U)$, and by Proposition 56,$w \notin \mathsf{BSh}(U)$. Hence $w \in \mathsf{iSh}(U)$.

  i. If $v_i \in \mathsf{BSh}(U)$, then both $w_i, v_i \in \mathsf{BSh}(\mu(U))$, and boundary share components unaffected.

  ii. If $v_i \in \mathsf{iSh}(U)$, then $v_i \in \mathsf{iSh}(\mu(U))$ and $w_i \in \mathsf{bSh}(\mu(U))$, while no change affects boundary share components.

  iii. If $v_i \in \mathsf{bSh}(U)$, then $v_i, w_i \in \mathsf{bSh}(\mu(U))$, while no change affects boundary share components.

  Hence, here we observe $\Delta\mathsf{iSh}(\mu) = -(h+1)$ and $\Delta\mathsf{BShC}(\mu) = 0$.

We conclude summing up the variations for the three portions of R so far considered:

$$\begin{aligned}\Delta\mathsf{iSh}(\mu) &= -h - 5 + h \times (\#\mathcal{E}(B) - 1) - (h+1) \\ &= h \times \#\mathcal{E}(B) - 3h - 6 \qquad (170)\\ \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) &= h \times \#\mathcal{E}(B) + 2h + 4 + \\ &\quad - \big(h \times \#\mathcal{E}(B) - 3h - 6\big) \\ &= 5h + 10 \qquad (171)\\ \Delta\mathsf{BShC}(\mu) &= 0 \qquad (172)\end{aligned}$$

4. Rule $(d!)$. Let R as in Figure 5e, and let $l, \mu(l)$ respectively be the lift of R and its residual. Recall that by definition u cannot belong to $\mathsf{bSh}(U)$.

a. If $l \in \mathsf{bLft}(U)$, then by definition also $\mu(l) \in \mathsf{bLft}(\mu)$. Moreover, we have $v_0 \notin \mathsf{Sh}(U)$ and $w \in \mathsf{BSh}(U)$. We separate two sub-cases about u.

i. If $u \in \mathsf{iSh}(U)$, then in the reduct, $w \notin \mathsf{Sh}(\mu(U))$ and $u \in \mathsf{BSh}(\mu(U))$. Therefore $\Delta\mathsf{iSh}(\mu) = -1 - 2 = -3$, and $\Delta\mathsf{BShC}(\mu) = 0$.

ii. If $u \in \mathsf{BSh}(U)$, then $w \notin \mathsf{Sh}(\mu(U))$ and $u \in \mathsf{BSh}(\mu(U))$. Observe that u belongs to a new boundary share component of $\mu(U)$. Therefore $\Delta\mathsf{iSh}(\mu) = 0 - 2 = -2$, and $\Delta\mathsf{BShC}(\mu) = 1$.

Summarising:

$$\begin{aligned}\Delta\mathsf{iSh}(\mu) &= -3 + \Delta\mathsf{BShC}(\mu) \qquad (173)\\ \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) &= 0 - (-3 + \Delta\mathsf{BShC}(\mu)) = 3 - \Delta\mathsf{BShC}(\mu) \qquad (174)\\ \Delta\mathsf{BShC}(\mu) &\in [0,1] \qquad (175)\end{aligned}$$

b. If $l \notin \mathsf{bLft}(U)$, then also $\mu(l) \notin \mathsf{bLft}(\mu(U))$. Moreover, $w \notin \mathsf{iSh}(U)$. There are then only three sub-cases we need to consider about w.

i. If $w \notin \mathsf{Sh}(U)$, which means that l is master, then $v_0, u \notin \mathsf{Sh}(U)$. Hence trivially, $v_0, u \notin \mathsf{Sh}(\mu(U))$.

ii. If $w \in \mathsf{bSh}(U)$, then by definition of pseudo-boundary, there exists a path $\pi_i : x_i \rightsquigarrow w$ (where it may be the case that $x_i = v_0$) such that $x_i \in \mathsf{iSh}(U)$ while $x'_i \in \mathsf{bSh}(U)$ for every $x'_i \neq x_i$ in π_i. In the residual we have x_i, x'_i unchanged with respect to the share. Moreover, $w \in \mathsf{iSh}(\mu(U))$, while $u \in \mathsf{BSh}(\mu(U))$.

iii. If $w \in \mathsf{BSh}(U)$, then $w \in \mathsf{BSh}(\mu(U))$. We need to distinguish two sub-cases about u, which by definition cannot belong to $\mathsf{bSh}(U)$.

A. If $u \in \mathsf{iSh}(U)$, then in the residual we observe that $w \in \mathsf{iSh}(\mu(U))$, while $u \in \mathsf{bSh}(\mu(U))$.

B. If $u \in \mathsf{BSh}(U)$, then trivially $u \in \mathsf{BSh}(\mu(U))$.

In all the three sub-cases, we accounted no variation in the metrics.

$$\begin{aligned}\Delta\mathsf{iSh}(\mu) &= 0 \qquad (176)\\ \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) &= 0 \qquad (177)\\ \Delta\mathsf{BShC}(\mu) &= 0 \qquad (178)\end{aligned}$$

5. Rules $(d\overset{+}{\multimap}), (d\overset{+}{\multimap}), (d?), (r\overset{-}{\multimap}), (r?)$. The analysis of $\Delta\mathsf{iSh}(\mu)$, $\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu)$ and $\Delta\mathsf{BShC}(\mu)$ is very similar to that of case 4, thus omitted to avoid pedantry.

6. Rule (a). Let R be as in Figure 5b, and let l, l' be the two lifts.

a. If $l, l' \in \mathsf{bLft}(U)$, then by definition, we have $u_0, v_0 \notin \mathsf{Sh}(U)$, while $w \in \mathsf{BSh}(U)$. After the reduction, the only residual vertex is $u_0 \notin \mathsf{Sh}(\mu(U))$. Hence,

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{179}$$
$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 0 \tag{180}$$
$$\Delta\mathsf{BShC}(\mu) = -1 \tag{181}$$

b. Otherwise $l, l' \notin \mathsf{bLft}(U)$, and we consider three sub-cases depending on v_0, which by definition cannot belong to $\mathsf{iSh}(U)$.

i. If $v_0 \notin \mathsf{Sh}(U)$, then also $u_0 \notin \mathsf{Sh}(U)$, which means that l, l' are master lifts. Trivially, we have $u_0 \notin \mathsf{Sh}(\mu(U))$.

ii. If $v_0 \in \mathsf{BSh}(U)$, then it must be the case that $u_0 \in \mathsf{BSh}(U)$ as well. Therefore $u_0 \in \mathsf{BSh}(\mu(U))$. Also, there exists $c \in \mathsf{BShC}(U)$ such that $u_0, w, v_0 \in c$ if and only if there exists $c' \in \mathsf{BShC}(U)$ such that $u_0 \in c$.

iii. If $v_0 \in \mathsf{bSh}(U)$, then by definition of pseudo-boundary, there exists a path $\pi_i : x_i \rightsquigarrow w$ (where it may be the case that $x_i = u_0$) such that $x_i \in \mathsf{iSh}(U)$ while $x'_i \in \mathsf{bSh}(U)$ for every $x'_i \neq x_i$ in π_i. In the residual we have x_i, x'_i unchanged with respect to the share positioning.

Therefore,

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{182}$$
$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 0 \tag{183}$$
$$\Delta\mathsf{BShC}(\mu) = 0 \tag{184}$$

7. Rule (s). Let R be as in Figure 5b, and let l, l' be the two lifts.

a. If $l, l' \notin \mathsf{bLft}(U)$ but $l, l' \in \mathsf{bLft}(\mu(U))$, then w necessarily belongs to $\mathsf{Sh}(U)$, but not to $\mathsf{iSh}(U)$. Before considering two sub-cases about w, we can already remark that, by hypotesis, one new positive boundary lift is introduced by μ.

i. If $w \in \mathsf{bSh}(U)$, then there exists $\pi_i : x_i \rightsquigarrow u_0$ (where possibly $x_i = u_0$) such that $x_i \in \mathsf{iSh}(U)$ while $x'_i \in \mathsf{bSh}(U)$ for every $x'_i \neq x_i$ in π_i. In the reduct we have $x_i, x'_i \in \mathsf{BSh}(\mu(U))$, while $z_{0_0} \notin \mathsf{Sh}(\mu(U))$.

ii. Otherwise $w \in \mathsf{BSh}(U)$, which implies that both $u_0, v_0 \in \mathsf{BSh}(U)$. In the reduct we still have $u_0, v_0 \in \mathsf{BSh}(\mu(U))$, while $z_{0_0} \notin \mathsf{Sh}(\mu(U))$. Moreover, while $u_0, v_0 \in c \in \mathsf{BShC}(U)$, after the reduction $u_0 \in c$ and $v_0 \in d$ where $c, d \in \mathsf{BShC}(\mu(U))$ and $c \neq d$.

Hence:

$$\Delta\mathsf{iSh}(\mu) = -1 + \Delta\mathsf{BShC}(\mu) \tag{185}$$
$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 1 - \Delta\mathsf{BShC}(\mu) \tag{186}$$
$$\Delta\mathsf{BShC}(\mu) \in [0, 1] \tag{187}$$

b. Otherwise, observe that if l, l' cannot both belong to $\mathsf{bLft}(U)$, since that would mean that $\ell(l) = \ell(l')$, which would imply that R is an (s)-redex, contradicting our hypothesis. Moreover, by definition we cannot have one interior and one boundary lift. Therefore we have only three possible cases.

i. If $w \notin \mathsf{Sh}(U)$, which means that l, l' are both master lifts, then we trivially have $u_0, w, v_0 \notin \mathsf{Sh}(U)$ and $u_0, z_{0_0}, v_0 \notin \mathsf{Sh}(\mu(U))$.

ii. If $w \in \mathsf{bSh}(U)$, which means that l, l' are both interior lifts, then we have $v_0 \in \mathsf{bSh}(U)$. Moreover, there exists $\pi_i : x_i \rightsquigarrow u_0$ (where possibly $x_i = u_0$) such that $x_i \in \mathsf{iSh}(U)$ while $x_i' \in \mathsf{bSh}(U)$ for every $x_i' \neq x_i$ in π_i. In the reduct we have $x_i', z_{0_0} \in \mathsf{bSh}(\mu(U))$, and $x_i \in \mathsf{iSh}(\mu(U))$, while no change can affect boundary share components.

iii. If $w \in \mathsf{Sh}(U)$, which means that one lift, say the positive l, is boundary, and the other, say the negative l', is master. The dual case is omitted for the sake of conciceness. By such assumption and by definition, we then have $u_0 \notin \mathsf{Sh}(U)$, while $w, v_0 \in \mathsf{BSh}(U)$. After the reduction, we obtain $u_0, z_{0_0} \notin \mathsf{Sh}(U)$, while $v_0 \in \mathsf{BSh}(U)$. Moreover, the number of boundary share components is unchanged.

We conclude by summarising:

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{188}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 0 \tag{189}$$

$$\Delta\mathsf{BShC}(\mu) = 0 \tag{190}$$

8. Rule (m). Let R be as in Figure 5h, and l its lift.

a. If $l \in \mathsf{bLft}(U)$, then $u_i \in \mathsf{BSh}()$, while $v_0, w \notin \mathsf{Sh}(U)$. After μ, we obtain no change with respect to the share positioning. On the other hand, $\{u_i\} \in \mathsf{BShC}(U)$, which is erased by μ. Therefore,

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{191}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 0 \tag{192}$$

$$\Delta\mathsf{BShC}(\mu) = -1 \tag{193}$$

b. Otherwise, we distinguish two sub-cases about u_0, which by our assumption cannot belong to $\mathsf{iSh}(U)$, nor to $\mathsf{BSh}(U)$.

i. If $u_i \notin \mathsf{Sh}(U)$, then trivially also $v_0, w \notin \mathsf{Sh}(U)$. In the reduction u_i is erased, and we have no change about internal share or its boundaries.

ii. If $u_i \in \mathsf{bSh}(U)$, then $w \notin \mathsf{Sh}(U)$. Also, there exists $\pi_i : x_i \rightsquigarrow v_0$ (where it may be the case that $x_i = v_0$) such that $x_i \in \mathsf{iSh}(U)$ while $x_i' \in \mathsf{bSh}(U)$ for every $x_i' \neq x_i$ in π_i. Now, in the residual we have that x_i, x_i' are unchanged with respect to the share positioning. Hence, again there is no change about internal share or its boundaries.

$$\Delta\mathsf{iSh}(\mu) = 0 \tag{194}$$

$$\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) = 0 \tag{195}$$

$$\Delta\mathsf{BShC}(\mu) = 0 \tag{196}$$

◂

A.4.2 L-contexts and unshared simulation

Things get more interesting, now. We study the effects of the unfolding and the simulation on sharing contexts in the unshared graph, finding a strong invariance. Consider a set V of vertices in a UG-graph such that they all belong to the unfolding of a same vertex in a ASG-graph. If in the l-context of $v \in V$ appears $x_{i:k}$, then we know that for any $0 \leq j \leq k$ there exists a unique $v' \in V$ whose l-context differs from that of v only for $x_{i:k}$, where instead appears $x_{j:k}$. This essentially means that the set V is in a bijection with all the possible variations to indices that one can perform on their sharing l-context. Thanks to this very strong property, we can easily obtain the proof of the fact that every vertex in a ASG-graph has a unique master vertex in its unfolding (Lemma 38), and similarly that, for every $k+1$-ary mux in a ASG-graph, there are exactly k boundary lifts in its unfolding (Lemma 64).

▸ **Proposition 63** (Unfolding and context permutations). *Let $\sigma \hookrightarrow \mu$ a pair of simulating reductions such that $N \xrightarrow{\sigma}{}^{*}_{\mathsf{ASG}} G$, and $N \xrightarrow{\mu}{}^{*}_{\mathsf{UG}} U$, where $G \hookrightarrow U$. Let $u \hookleftarrow v$ for some $u \in V(U), v \in V(G)$ such that $a \cdot b \cdot c = \mathfrak{c}(u)(n)$ stable, for some $a, b, c \in \mathfrak{C}^*$. If $b = x_{i:k}$ (or $\overline{x_{i:k}}$), then for any $0 \le i' \le k$ there is a unique $u' \hookleftarrow v$ such that:*

1. *$\mathfrak{c}(u')(n) = a \cdot b' \cdot c$, with $b' = x_{i':k}$ (or $\overline{x_{i':k}}$, respectively); and*
2. *$\mathfrak{c}(u')(n') = \mathfrak{c}(u)(n')$, for any $n' \neq n$.*

Proof of Proposition 63. We go by induction on the length of the ASG-reduction $\bar{\sigma}$ such that $G = \bar{\sigma}(N)$ for some levelled proof-net N. The base case is trivial, because when $|\bar{\sigma}| = 0$ we have $G = N$ and since $G \hookrightarrow U$ also $U = N$. Therefore, for any $s \in V(G)$ we have $\mathfrak{c}(s) = 1$ and the claim vacuously holds. So assume otherwise and in particular: let $G = \sigma(G')$ for some ASG-reduction step σ on a redex S such that $G' = \bar{\sigma}'(N)$; let $U' \hookleftarrow G'$; and let $\bar{\mu} \hookleftarrow \sigma$ for some UG-reduction sequence $\bar{\mu}$ on a set of redexes $\bar{M}$ such that $U = \bar{\mu}(U') = \bar{\mu}(\bar{\mu}'(N))$. Finally, take $s \in V(G)$ and $t \hookleftarrow s$ such that $\mathfrak{c}(t)(n) = a \cdot b \cdot c$ where $b = x_{l:m}$ or $\overline{x_{l:m}}$, for some $n, l, m \in \mathbb{N}$, and some $a, x_{l:m}, c \in \mathfrak{C}^*$. Finally let $0 \le l' \le m$ be the index for which we want to prove the claim.

1. If $s \notin \mathsf{int}(\sigma(S))$, then by definition of reduction $\sigma^{-1}(s) = s$. Moreover, for any $M \in \bar{M}$ we also have $t \notin \bar{\mu}(M)$, as per definition of unfolding. Therefore, in accordance with invariance Lemma 62, $\mathfrak{c}(t) = \mathfrak{c}\left(\bar{\mu}^{-1}(t)\right)$. So let $\bar{\mu}^{-1}(t_l) \hookleftarrow \bar{\sigma}^{-1}(s)$ be the unique vertex of U' such that the claim holds by IH. Now, since $\mathfrak{c}\left(\bar{\mu}^{-1}(t_l)\right) = \mathfrak{c}(t_l)$, we trivially conclude that the claim still holds.
2. If $s \in \mathsf{int}(\sigma(S))$, then let $M \in \bar{M}$ such that $t \in M$, and let $\mu \in \bar{\mu}$ be the reduction step on M. We proceed with a case analysis on the kind of the redex S of σ.
   a. Rules $(!), (\multimap), (a), (m)$. Absurd: by inspection of the definition of the redexes, we verify that $\mathsf{int}(\sigma(S)) = \varnothing$, contradicting the hypothesis of $s \in \mathsf{int}(\sigma(S))$.
   b. Rule $(d!)$. Let S as in Figure 5e, and, since $\mathsf{int}(\sigma(S)) = \{z_0, \dots, z_k\}$, let $s = z_i$. Now, assume vertex names in M are as in S, but with the prime symbol, and let $t = z'_{i'}$. Observe that in $\mu(M)$, as per definition of context assignment, we have $\mathfrak{c}(v_{i'}) = \mathfrak{c}(z_{i'})$. But $\mathfrak{c}(v_{i'})$ belongs to $\mathsf{iface}(M)$, where by Lemma 62 it must have the same levelled context. Now, since $v_i \hookrightarrow v_{i'}$, let $v''_{i''}$ be the unique vertex in U' such that $v_i \hookrightarrow v''_{i''}$ and it satisfies the claim, i.e. $\mathfrak{c}(v''_{i''})(n) = a \cdot x_{l':m} \cdot c$, while $\mathfrak{c}(v''_{i''})(n') = \mathfrak{c}(v'_{i'})(n')$, for any $n' \neq n$. So, let M'' be the redex containing $v''_{i''}$, and μ'' be its reduction step. Since it belongs to the $\mathsf{iface}(M'')$, we find $v''_{i''}$ also in $\mu''(M'')$ with the same levelled-context. Consider $z''_{i''}$ in $\mu''(M'')$ and verify not only that $z_i \hookrightarrow z''_{i''}$, but also that, as previously remarked, $\mathfrak{c}(v''_{i''}) = \mathfrak{c}(z''_{i''})$. Hence the claim.
   c. Rules $(d?), (d\overset{+}{\multimap}), (d\overset{-}{\multimap}), (s), (r\overset{-}{\multimap}), (r?)$. The argument detailed in case 2b can be applied here with only minor changes.
   d. Rule (tD). Let S as in Figure 6, but let d be the level of !-link, and observe that $\mathsf{int}(\sigma(S)) = \bigcup_{1 \le i \le h} B_i$, where B_i denotes the i-copy of the interior of the box B. Let the names of vertices and boxes in M be as in S, but with prime symbols. So let $t \in B'_{i'}$ and consider $\sigma^{-1}(t)$. Assume the downward crossing of the i-th lift above $B'_{i'}$ has context $!^n e$, Also, let $\mathfrak{c}\left(\mu^{-1}(t)\right)(d) = f \cdot g$, where $f = \mathfrak{c}(v_i)$. Then, by definition of

context assignment, we first observe the following.

$$\mathfrak{c}(t)(n') = \begin{cases} \mathfrak{c}\left(\sigma^{-1}(t)\right) & \text{if } d' \neq d; \\ f \cdot e \cdot g & \text{if } d' = d. \end{cases} \tag{197}$$

Now, we observe that we can apply IH on $\mu^{-1}(t)$. Namely, for any $n \neq d$ and for any $x_{l:m} \neq e$ (or its negation), appearing in $\mathfrak{c}\left(\mu^{-1}(t)\right)$ at the n-th context, we know there exists a unique vertex t' in U, such that $\sigma^{-1}(s) \hookrightarrow t'$ and for which: in $\mathfrak{c}(t')(n)$ is obtained from $\mathfrak{c}(t)$ substituting $x_{l:m}$ for $x_{l':m}$, while $\mathfrak{c}(t')(n') = \mathfrak{c}\left(\mu^{-1}(t)\right)(n')$, for any $n' \neq n$.
Now observe that t' belongs to a box B'' in a redex M', so let μ' be its reduction step, which by definition of unfolding belongs to $\bar{\mu}$. Now, let $t'' \in \mu'(t')$ be the vertex belonging in the i-th copy of the box B'', and verify that it satisfies the claim.

◂

Proof of Lemma 38. Immediate from Proposition 63 by fixing $i' = 0$ in the claim. ◂

▸ **Lemma 64** (Arities of muxes and cardinalities of boundary lifts)**.** *Let $N \in EPN$, $U \in UG$ and $G \in ASG$ such that $N \xrightarrow{\bar{\sigma}}^*_{ASG} G$ and $N \xrightarrow{\bar{\mu}}^*_{UGR} U$ with $\bar{\sigma} \hookrightarrow \bar{\mu}$. For any mux $m \in L(G)$ of arity $k+1$, let $L = \{l \in \mathsf{bLft}(U) \mid m \hookrightarrow l\}$. Then $\#L = k$.*

Proof of Lemma 64. Let $m = (u_0, \dots u_k \;|\!\ast\rangle\; z)$ and consider the set $L' \subseteq L(U)$ of any lift whose premiss is the master copy of u_i, for some $0 \leq i \leq k$. Recall that, thanks to Lemma 38, we know that for any i the master copy is unique, so $\#L' = k+1$. Now by definition of the unfolding relation, there is a sharing morphism between U and G which is connection-preserving and surjective (cf. Definition 21). Therefore, the master copy of z must be the conclusion of a lift $l_m \in L'$, which consequently is a master lift. Now let $L_b = L' \smallsetminus l_m$ and notice that it contains all and only the boundary lifts that are unfolding of m. Indeed, for every $l_i \in L_B$ different than l_m, we have $u_i \notin \mathsf{Sh}(U)$ while $z_i \in \mathsf{Sh}(U)$. Thus $L_b = L$ and trivially $\#L_B = k$. ◂

A.4.3 Correctness of the unshared cost for sharing reduction

Proof of Lemma 39. We go by induction on $|\bar{\sigma}|$. The base case is trivial, because when $|\bar{\sigma}| = 0$ we have $G = N$ and since $G \hookrightarrow U$ also $U = N$. This means that $\mathcal{C}_{\mathsf{ASG}}(\bar{\sigma}) = \mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) = 0$, so the claim vacuously holds. So assume $|\bar{\sigma}| > 0$. Let $G = \sigma(G')$ for some ASG-reduction step σ on a redex S such that $G' = \bar{\sigma}'(N)$; let $U' \hookleftarrow G'$; and let $\bar{\mu} \hookleftarrow \sigma$ for some UG-reduction sequence $\bar{\mu}$ on a set of redexes $\bar{M}$ such that $U = \bar{\mu}(U') = \bar{\mu}(\bar{\mu}'(N))$. By inductive hypothesis we also assume $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}') = \mathcal{C}_{\mathsf{ASG}}(\bar{\sigma}')$, so we need to prove that $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) = \mathcal{C}_{\mathsf{ASG}}(\sigma)$.

1. Suppose that S does not contain muxes, i.e. the rule of S is one of the following: $(\overset{+}{-}\!\circ), (!), (t)$. Then by Lemma 38, let $M \in \bar{M}$ be the unique redex such that for any $v' \in M$ and any $v \in S$, if $v \hookrightarrow v'$ then v' is the master copy of v. Now, let μ be the reduction of M and $\bar{\mu}''$ be the reduction of $\bar{M} \smallsetminus M$. Now, observe that by Definition 19 of $\mathcal{C}_{\mathsf{ASG}}$ (see in particular Table 1b) and Definition 37 of $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}$ (see in particular Table 2) we have $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\mu) = \mathcal{C}_{\mathsf{ASG}}(\sigma)$, while $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}'') = 0$. Therefore $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) = \mathcal{C}_{\mathsf{ASG}}(\sigma)$.
2. If S contains muxes, and is not a (s) rule, then let $\bar{M}'' \subset \bar{M}$ be the set of any redex in $\bar{M}$ whose lifts are boundary. First, observe that, again by Definition 19 and 37 (cf. Table 1b and 2), for any reduction step μ'' on a redex of $\bar{M}''$, we have $\mathcal{C}_{\mathsf{ASG}}(\sigma) = k \times \mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu''})$, where $k+1$ is the number of premisses of the mux of S. Conversely, for any reduction

step μ''' on a redex of $\bar{M} \smallsetminus \bar{M}''$, as per definition, $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu'''}) = 0$. But by Lemma 64 we know that $|\bar{M}''| = k$. Thus, $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) = \mathcal{C}_{\mathsf{ASG}}(\sigma)$.

3. Otherwise S is a (s) rule. Let $k+1$ and $l+1$ be the number of premisses of the positive and negative mux of S, respectively. Observe that by definition of reduction (cf. Figure 5b) in $\sigma(S)$ there are $k+1$ negative lifts with arity $l+1$, and $l+1$ positive lifts with arity $k+1$. By Lemma 64 this implies that $\bar{M}$ contains k positive boundary lifts and l negative boundary lifts. Since by definition of boundary lift, there cannot be two boundary lifts in a (s) redex, the boundary lifts of $\bar{M}$ belong to different redexes. Now, again per Lemma 64, it must be the case that $\bar{\mu}(\bar{M})$ contains $(k+1) \times l$ negative lifts, and $(l+1) \times k$ positive lifts. But this means that $\bar{\mu}$ introduces $k \times l$ negative lifts and $k \times l$ positive lifts, i.e. there are $k \times l$ redexes in $\bar{M}$ made of two pseudo-boundary lifts which become boundary when swapped. (This happens when the levelled context of premiss of the positive lift is 1 everywhere but at the level of the negative lift, where the context is identical to the variable occurrence of the and the negative lift.) Since $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}$ assign a unitary cost of such reduction, and 0 otherwise, we obtain that $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) = k \times l = \mathcal{C}_{\mathsf{ASG}}(\sigma)$. ◂

A.4.4 Comparison of unshared costs

Proof of Lemma 40. Given a step μ of $\bar{\mu}$, we compare the possible values of $\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\mu) + \mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\mu)$ and $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\mu)$. A mere inspection of Lemma 35 and Definition 37 and, in particular, a comparison of the corresponding cases of Table 2 and Table 2, allow to immediately verify the inequation. ◂

▸ **Lemma 65** (Redundant boundary share components). *Given $U \in UG$ and $v \in \mathsf{BLSh}(U)$, let $L : u \sim v$ be the boundary lift chain of v and let $\bar{\mu}$ be a reduction sequence on L. Then, the number of all (s)-step $\mu \in \bar{\mu}$ such that $\Delta\mathsf{BShC}(\mu) = 1$ is less or equal to $\partial(U)$.*

Proof of Lemma 65. By hypothesis, $u \rightsquigarrow v$ or $v \rightsquigarrow v$. If both hold, then $u = v$ and the claim is trivially verified, because $L = \varnothing$. Otherwise, since the arguments of the two cases are identical, we shall assume $u \rightsquigarrow v$. Let $\bar{\mu}_{cs}$ be the set of any (s)-step μ in $\bar{\mu}$ such that $\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\mu) = 1$.

We will now prove that for any level n there may be at most one step $\mu \in \bar{\mu}_{cs}$. This implies the claim, because in the worst case L contains lifts of any level $l \le \partial(U)$.

For the sake of contradiction, assume otherwise that L contains: a $|{+}\rangle^{n}$-link l, a $|{+}\rangle^{n'}$-link l', and $|{-}\rangle^{n''}$-link l'' such that there are two steps in $\bar{\mu}$ swapping both l, l' with l''. (The dual situation with one $|{+}\rangle$-link and two $|{-}\rangle$-link is identical, hence omitted.) Let $\mathsf{CID}(l) = a$, $\mathsf{CID}(l') = b$, and $\mathsf{CID}(l'') = c$. Then, there are some l-contexts $\gamma_1, \gamma_2, \gamma_3$ such that:

$$\mathfrak{c}\,(v) = \delta \cdot \underbrace{\gamma_1 \cdot !^{n} a \cdot \gamma_2 \cdot !^{n'} b \cdot \gamma_3}_{\gamma_0} \cdot !^{n''} \overline{c} \cdot \gamma_4, \tag{198}$$

where γ_0 is the l-context of the downward path π that goes from u to the conclusion of l''. Now, L contains only $|{*}\rangle$-links, therefore any reductions in $\bar{\mu}$ is necessarily an (s)- or an (a)-rule. By hypothesis there is no (a)-rule involving l, l' or l'', hence π is long enough for any reduction step of $\bar{\mu}$. Therefore the context of π is invariant under any reduction preceding the swap between l' and l''. For such (s)-step we hypothesised that in the redex the lifts are interior, while in the reduct they are boundary. Thus $\mathfrak{c}\,(\gamma_0)\,(n') = b$ while $\mathfrak{c}\,(\gamma_0)\,(m) = 1$ for any $m \ne n'$. But given the presence of the weight of l, i.e. $!^{n} a$, this implies that $\mathfrak{c}\,(\gamma_2)\,(n) = \overline{a}$, while $\mathfrak{c}\,(\gamma_2)\,(m) = 1$, for any $m \ne n$. Therefore that π contains a $|{-}\rangle$-link which annihilate with l. Absurd. ◂

Proof of Lemma 41. Let $\bar{\mu}$ a UGR-reduction sequence. We partition the reduction steps of $\bar{\mu}$ into three sets depending on their action on BShC: erasure, creation with swap rules, the rest, which may cause creations. Formally we define:

$\bar{\mu}_e$ the set of any step μ of $\bar{\mu}$ such that $\Delta\mathsf{BShC}(\mu) = -1$, hence μ is of kind $(a),(m)$;
$\bar{\mu}_{cs}$ the set of any step μ of $\bar{\mu}$ such that $\Delta\mathsf{BShC}(\mu) = 1$ and μ is of kind (s);
$\bar{\mu}_{co}$ the set of any step μ of $\bar{\mu}$ that does not belong to $\bar{\mu}_e$ or $\bar{\mu}_{cs}$.

Hence, by construction:

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}) = \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_e) + \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs}) + \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{co}), \tag{199}$$

so we can separately discuss the three addends.

1. $\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_e)$. We observe that, since $\#\mathsf{BShC}(\cdot)$ is always non negative, the number of erasures cannot outnumber the creations. Therefore,

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_e) \leq \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs}) + \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{co}). \tag{200}$$

2. $\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs})$. Let $\bar{\mu}'_{cs}$ be a maximal subsequence of $\bar{\mu}$ such that, if $\bar{\mu} = \bar{\nu}\bar{\mu}'_{cs}\bar{\omega}$ for some reduction sequences $\bar{\nu}, \bar{\omega}$, then any step of $\bar{\mu}'_{cs}$ acts on the boundary lift chain L of every $v \in \mathsf{BLSh}(\bar{\nu}(U))$. Now, by Lemma 65 we have that the normalisation of L entail and increase of $\#\mathsf{BShC}$ of at most $\partial(U)$.
Moreover, we can easy observe that by definition, for any $U' \in \mathsf{UG}$ $\#\mathsf{BLSh}(U') \leq \#\mathsf{iSh}(U')$. Hence, $\#\mathsf{BLSh}(\bar{\nu}(U)) \leq \#\mathsf{iSh}(\bar{\nu}(U))$. Therefore

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}'_{cs}) \leq \partial(U) \times \#\mathsf{iSh}(\bar{\nu}(U)), \tag{201}$$

and, more simply:

$$\leq \partial(U) \times \overline{\mathcal{C}}_{\mathsf{UG}}^{\mathsf{EPN}}(\bar{\mu}) \tag{202}$$

Now, observe that the distinct sub-sequences of $\bar{\mu}$ defined as $\bar{\mu}'_{cs}$ are at most as many as the number of steps $\mu \in \bar{\mu}$ such that $\mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu) = 0$. Therefore:

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs}) \leq \partial(U) \times \mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu) \times \overline{\mathcal{C}}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu), \tag{203}$$

where, since $\mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu) \leq \overline{\mathcal{C}}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu)$, we can loosen and simplify as:

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs}) \leq \partial(U) \times (\overline{\mathcal{C}}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu))^2. \tag{204}$$

3. $\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{co})$. Recall from the account provided by Lemma 35 in Table 2 that, for any step $\mu \in \bar{\mu}_{co}$, $\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\mu) \leq \mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\mu)$. Therefore,

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{co}) \leq \mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\bar{\mu}_{co}) \tag{205}$$
$$\leq \mathcal{C}_{\mathsf{UG}}^{\mathsf{EPN}}(\bar{\mu}) \tag{206}$$
$$\leq \overline{\mathcal{C}}_{\mathsf{UG}}^{\mathsf{EPN}}(\bar{\mu}). \tag{207}$$

Now we go back to (199) and substitute first (200):

$$\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}) \leq 2 \times \left(\mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{cs}) + \mathcal{C}_{\mathsf{UG}}^{\mathsf{BShC}}(\bar{\mu}_{co})\right) \tag{208}$$

and then (204) and (207)

$$\mathcal{C}^{\mathsf{BShC}}_{\mathsf{UG}}(\bar{\mu}) \leq 2 \times \left(\partial(U) \times \left(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right)^2 + \overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right) \tag{209}$$

$$\leq 2 \times \overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu}) + 2 \times \partial(U) \times \left(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right)^2 . \tag{210}$$

◂

Proof of Theorem 1. By definition of $\rightarrowtail$, there exists $U \in \mathsf{UG}$ such that $G \hookrightarrow U \mapsto N$ and there exists an unfolded unshared reduction $\bar{\mu} : N \to^*_{\mathsf{UG}} U$. On one side we know that $\mathcal{C}_{\mathsf{EPN}}(\bar{\rho}) = \overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})$ (Fact 34), and on the other $\mathcal{C}_{\mathsf{ASG}}(\bar{\sigma}) = \mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu})$ (Lemma 39). Now, as per Lemma 41 and Lemma 40, there exists a quadratic function q such that: $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) \leq \mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu}) + q\left(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right)$. But since $\mathcal{C}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu}) \leq \overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})$, by definition we have $\mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) \leq q\left(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right)$, therefore $\mathcal{C}_{\mathsf{ASG}}(\bar{\sigma}) = \mathcal{C}^{\mathsf{ASG}}_{\mathsf{UG}}(\bar{\mu}) \leq q\left(\overline{\mathcal{C}}^{\mathsf{EPN}}_{\mathsf{UG}}(\bar{\mu})\right) = q\left(\mathcal{C}_{\mathsf{EPN}}(\bar{\rho})\right)$. ◂